Quantum Computing: Exercise Sheet 1

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Note: Unitarity conditions $UU^\dagger = I$ and $U^\dagger U = I$ are equivalent, so you only need to check one of them to verify that a given matrix U is unitary. This equivalence is not immediately obvious. It follows from a slightly more general fact that $AB = I$ and $BA = I$ are equivalent for any square matrices A and B. If you want to know why this is the case, see: http://math.stackexchange.com/questions/3852/if-ab-i-then-ba-i.

Note: Problems marked by (☠) are harder. Hints for some problems are provided on the last page.

Lecture 1: Bits and qubits

1. Which of the following are possible states of a qubit (for any $\theta \in \mathbb{R}$)?

   (a) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

   (b) $\frac{\sqrt{3}}{2} i|1\rangle - \frac{1}{2}|0\rangle$

   (c) $0.7|0\rangle + 0.3|1\rangle$

   (d) $0.8|0\rangle + 0.6|1\rangle$

   (e) $\cos\theta|0\rangle + i\sin\theta|1\rangle$

   (f) $\cos^2\theta|0\rangle - \sin^2\theta|1\rangle$

   (g) $\frac{1}{2}|0\rangle + \frac{i}{2}|0\rangle - \frac{i}{2}|1\rangle + \frac{1}{2}|1\rangle$

   For each valid state among the above, give the probabilities of observing $|0\rangle$ and $|1\rangle$ when the system is measured in the standard basis.

   What are the probabilities of the two outcomes when (a), (b), (e) are measured in the *Hadamard basis* $\{|+\rangle, |-\rangle\}$ where

   $$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)?$$

   Do the probabilities in case (e) depend on θ? Why?

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2. Locate the following six states on the Bloch sphere by computing their (x, y, z) coordinates:

   $$|0\rangle, |1\rangle, \qquad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \qquad \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle).$$

   How are the angles between these states related to the angles between their Bloch vectors?

Lecture 2: Review of linear algebra

3. Let A be an $n \times n$ matrix given in the computational basis and let $\langle i|$ denote the i-th standard basis row vector and $|j\rangle$ denote the j-th standard basis column vector. Show that $\langle i|A$ is the i-th row of A and $A|j\rangle$ is the j-th column of A. Show that $\langle i|A|j\rangle = A_{ij}$, the (i,j)-th entry of A.

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4. Verify that the inner product $\langle u|v\rangle$ on $\mathbb{C}^n$ satisfies the following two properties:

   - $\langle u|v\rangle = \big(\langle v|u\rangle\big)^\dagger = \overline{\langle v|u\rangle}$
   - $\langle u|u\rangle \geq 0$ and $\langle u|u\rangle = 0$ iff $|u\rangle = \mathbf{0}$ (the all-zeroes vector)

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5. Find the eigenvalues and associated eigenvectors of $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

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6. Show that any Hermitian matrix H can be written as $H = S + iA$ in a unique way where both S and A are real, S is symmetric ($S^\mathsf{T} = S$) while A is anti-symmetric ($A^\mathsf{T} = -A$).

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7. Let A be a matrix whose spectral decomposition is $A = \sum_k \lambda_k |u_k\rangle\langle u_k|$.

   (a) Show that $\lambda_k \in \mathbb{R}$ for all k if and only if A is Hermitian and that $|\lambda_k| = 1$ for all k if and only if A is unitary.

   (b) What is the spectral decomposition of $A^2 \equiv A \cdot A$?

   (c) Show that $\exp(A) \equiv \sum_{n=0}^{\infty} A^n/n!$ has spectral decomposition $A = \sum_k e^{\lambda_k} |u_k\rangle\langle u_k|$. Show that $\exp(-iA)$ is unitary if A is Hermitian.

   (d) Assume A is Hermitian *and* unitary. What are the eigenvalues of A? How does A^2 look like?

   Note: Part (c) explains why the solution of Schrödinger equation (see Postulate 2 in Lecture 3) corresponds to unitary evolution.

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8. Let $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ be the 2-dimensional identity matrix and $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ be the Hadamard matrix. Compute the matrices $I \otimes H$ and $H \otimes I$.

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9. (☠) Let $|\psi\rangle \in \mathbb{C}^n$ and $|\varphi\rangle \in \mathbb{R}^n$. How are the entries of $|\psi\rangle\langle\varphi|$ and $|\psi\rangle \otimes |\varphi\rangle$ related? Does there exist a fixed vector $|\Phi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ such that $\big(|\psi\rangle\langle\varphi| \otimes I\big)|\Phi\rangle = |\psi\rangle \otimes |\varphi\rangle$ for all $|\psi\rangle$ and $|\varphi\rangle$? (See Hint 1 on the last page.)

Lecture 3: Principles of quantum mechanics

10. Let $\{|u_1\rangle, \ldots, |u_n\rangle\}$ and $\{|v_1\rangle, \ldots, |v_n\rangle\}$ be any two orthonormal bases of $\mathbb{C}^n$. Let I denote the $n \times n$ identity matrix, i.e., the unique matrix such that $I|w\rangle = |w\rangle$ for all $|w\rangle \in \mathbb{C}^n$. Show that $\sum_{i=1}^{n} |u_i\rangle\langle u_i| = I$. Write down the matrix U that changes basis from $\{|u_1\rangle, \ldots, |u_n\rangle\}$ to $\{|v_1\rangle, \ldots, |v_n\rangle\}$. Verify explicitly that U is unitary.

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11. Express the matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the basis $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$.

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12. Verify that each of the Pauli matrices $\{I, X, Y, Z\}$ is Hermitian. Check that $P^2 = I$ for every $P \in \{I, X, Y, Z\}$. Why does this imply that Pauli matrices are unitary? Deduce from this the eigenvalues of the Pauli matrices. Given that all rules for quaternion multiplication can be derived from the identity $i^2 = j^2 = k^2 = ijk = -1$ inscribed into the Brougham Bridge in Dublin, what other property of Pauli matrices you need to verify to show that they are equivalent to quaternions? Verify this property.

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13. Verify that an orthogonal measurement is indeed a special case of a general measurement. More specifically, verify that the measurement operators of an orthogonal measurement satisfy the completeness relations and that the outcome probability and post-measurement state formulas agree with the general case.

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14. Assume that $|\Psi\rangle$ is an entangled two-qubit state and show that $|\Psi'\rangle = (U \otimes V)|\Psi\rangle$ is also entangled for any choice of single-qubit unitary transformations U and V.

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15. Any 2×2 unitary matrix U can be written as follows:

$$U = e^{i\varphi} \begin{pmatrix} c & -a \\ a^* & c^* \end{pmatrix}$$

where $\varphi \in \mathbb{R}$ and $a, c \in \mathbb{C}$ are such that $|a|^2 + |c|^2 = 1$. Verify that this matrix is indeed unitary.

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16. Let $|\Psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ be a two-qubit state, $a, b, c, d \in \mathbb{C}$. Show that $|\Psi\rangle$ is entangled if and only if $ad \neq bc$.

You need to prove two directions:

(a) Assume that $|\Psi\rangle$ is a product state and show that in such case $ad = bc$.

(b) (☠) Prove the converse: assume that $ad = bc$ and show that $|\Psi\rangle$ is a product state (see Hint 2 on the last page).

Based on this condition, can you propose some way of quantifying the "amount" of entanglement in a two-qubit state $|\Psi\rangle$?

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17. Verify that $|\Psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is entangled using the condition from the previous problem. You can also try to derive the same conclusion directly by first assuming that $|\Psi\rangle$ is a product state and then getting a contradiction.

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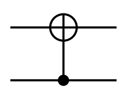
18. A two-qubit system is in the following state:

$$|\psi\rangle = \frac{1}{\sqrt{30}}(|00\rangle + 2i|01\rangle - 3|10\rangle - 4i|11\rangle)$$

The first qubit is measured in the standard basis and observed to be 1. What is the state of both qubits after the measurement? (See Hint 3 for a simple method.) What is the probability that a subsequent measurement of the second qubit in the standard basis will produce the outcome 1?

Lecture 4: Model of quantum computation

19. Write down the matrix representation for the upside down CNOT gate:



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20. Write down the Toffolli gate in matrix form.

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21. (☠) **No-cloning theorem**. The no-cloning theorem states that a quantum state $|\psi\rangle$ cannot be duplicated. That is, there is no single two-qubit unitary U such that

$$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$

holds for all $|\psi\rangle \in \mathbb{C}^2$ simultaneously. Prove this theorem.

Note: The same proof shows also that there is no linear (stochastic) map that can copy an unknown probability distribution. Both arguments boil down to showing that the following map is not linear:

$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \otimes \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$

Hint 4. Since U must work for all states, it must work in particular for the standard basis:

$$U|0\rangle|0\rangle = |0\rangle|0\rangle$$
$$U|1\rangle|0\rangle = |1\rangle|1\rangle$$

Derive a contradiction by choosing a state $|\psi\rangle \in \mathbb{C}^2$ such that $U|\psi\rangle|0\rangle \neq |\psi\rangle|\psi\rangle$.

Hint 3. Simply collect all terms that have $|1\rangle$ in the first qubit and normalize their sum.

Hint 2. Apply the matrix U from the previous problem on the first qubit.

Hint 1. Well done, you figured out how to read this! However, this is supposed to be a trick question, so no further hints on this one!