9: Viterbi Algorithm for HMM Decoding

Machine Learning and Real-world Data

Simone Teufel and Ann Copestake

Computer Laboratory
University of Cambridge

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Last session: estimating parameters of an HMM

- The dishonest casino, dice edition
- Two states: L (loaded dice), F (fair dice). States are hidden.
- You estimated transition and emission probabilities.
- Now let's now see how well an HMM can discriminate this highly ambiguous situation.
- We need to write a decoder.

Decoding: finding the most likely path

- Definition of decoding: Finding the most likely state sequence X that explains the observations, given this HMM's parameters.

$$\hat{X} = \operatorname*{argmax}_{X_0 \ldots X_{T+1}} P(X|O,\mu) =$$

$$\operatorname*{argmax}_{X_0 \ldots X_{T+1}} \prod_{t=0}^{T+1} P(O_t|X_t)P(X_t|X_{t-1})$$

- Search space of possible state sequences X is O(N^T); too large for brute force search.

Viterbi is a Dynamic Programming Application

(Reminder from Algorithms course)
We can use Dynamic Programming if two conditions apply:

- Optimal substructure property
  - An optimal state sequence $X_0 \dots X_j \dots X_{T+1}$ contains inside it the sequence $X_0 \dots X_j$, which is also optimal
- Overlapping subsolutions property
  - If both X_t and X_u are on the optimal path, with $u > t$, then the calculation of the probability for being in state X_t is part of each of the many calculations for being in state X_u.

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The intuition behind Viterbi

- Here's how we can save ourselves a lot of time.
- Because of the Limited Horizon of the HMM, we don't need to keep a complete record of how we arrived at a certain state.
- For the first-order HMM, we only need to record one previous step.
- Just do the calculation of the probability of reaching each state once for each time step.
- Then memoise this probability in a Dynamic Programming table
- This reduces our effort to $O(N^2 T)$.
- This is for the first order HMM, which only has a memory of one previous state.

Viterbi: main data structure

- Memoisation is done using a *trellis*.
- A trellis is equivalent to a Dynamic Programming table.
- The trellis is $N \times (T + 1)$ in size, with states j as rows and time steps t as columns.
- Each cell j, t records the Viterbi probability $\delta_j(t)$, the probability of the optimal state sequence ending in state s_j at time t:

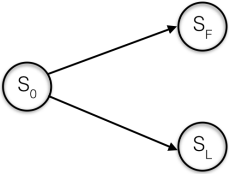
$$\delta_j(t) = \max_{X_0,\ldots,X_{t-1}} P(X_0 \ldots X_{t-1}, o_1 o_2 \ldots o_t, X_t = s_j | \mu)$$

Viterbi algorithm, initialisation

- The initial $\delta_j(1)$ concerns time step 1.
- It stores, for all states, the probability of moving to state s_j from the start state, and having emitted o_1.
- We therefore calculate it for each state s_j by multiplying transmission probability a_{0j} from the start state to s_j, with the emission probability for the first emission o_1.

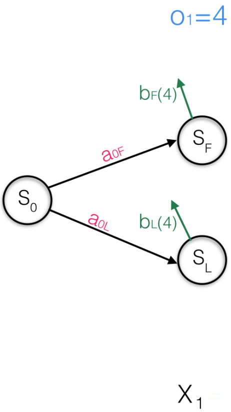
$$\delta_j(1) = a_{0j} b_j(o_1), 1 \leq j \leq N$$

Viterbi algorithm, initialisation

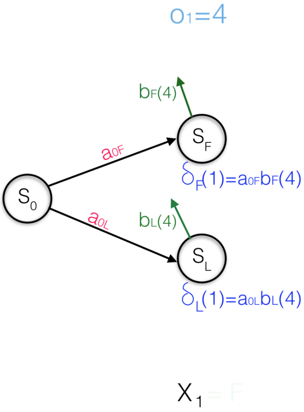


X_1

Viterbi algorithm, initialisation: observation is 4



Viterbi algorithm, initialisation: observation is 4

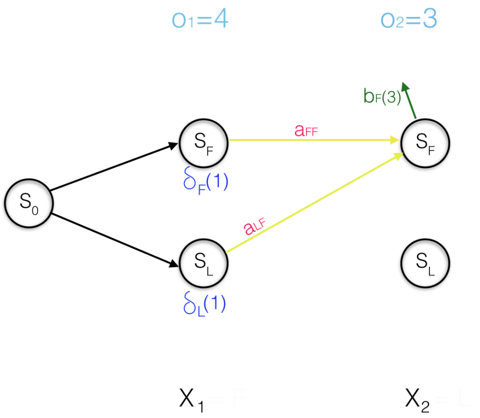


Viterbi algorithm, main step, observation is 3

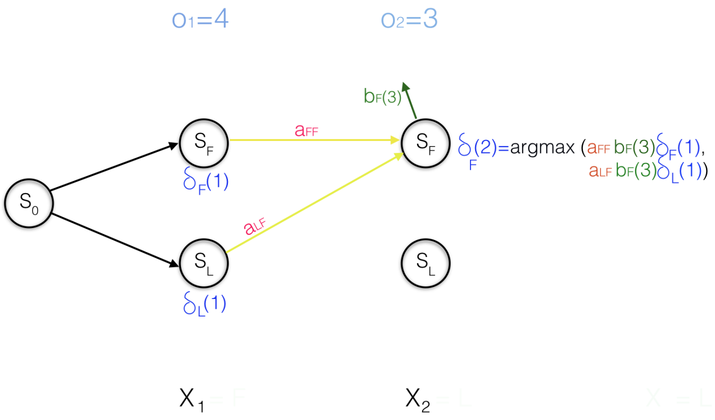
- $\delta_j(t)$ stores the probability of the best path ending in s_j at time step t.
- This probability is calculated by maximising over the best ways of transmitting into s_j for each s_i.
- This step comprises:
  - $\delta_i(t-1)$: the probability of being in state s_i at time $t-1$
  - a_{ij}: the transition probability from s_i to s_j
  - $b_i(o_t)$: the probability of emitting o_t from destination state s_j

$$\delta_j(t) = \max_{1 \leq i \leq N} \delta_i(t-1) \cdot a_{ij} \cdot b_j(o_t)$$

Viterbi algorithm, main step



Viterbi algorithm, main step



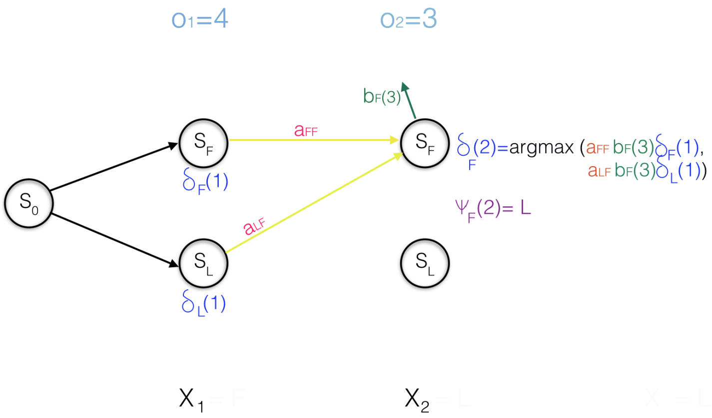
Viterbi algorithm, main step, ψ

- $\psi_j(t)$ is a helper variable that stores the $t-1$ state index i on the highest probability path.

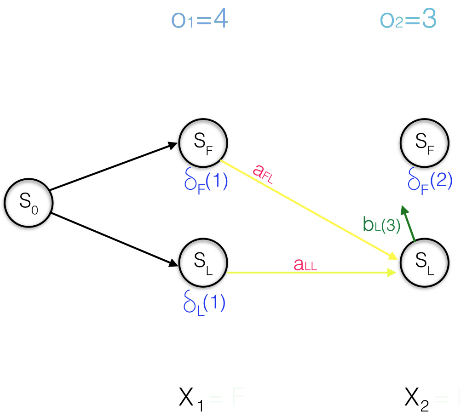
$$\psi_j(t) = \operatorname*{argmax}_{1 \leq i \leq N} \delta_i(t-1) a_{ij} b_j(o_t)$$

- In the backtracing phase, we will use ψ to find the previous cell in the best path.

Viterbi algorithm, main step

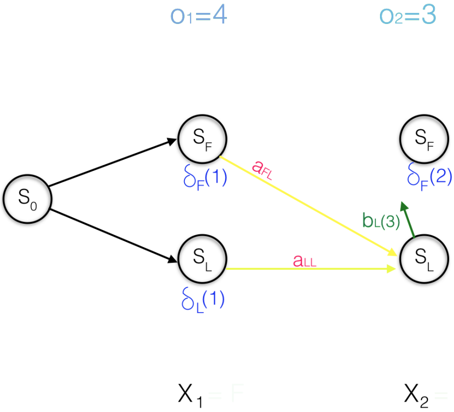


Viterbi algorithm, main step



$$\delta_L(2) = \text{argmax}\,(a_{FL}\, b_L(3)\, \delta_F(1),\; a_{LL}\, b_L(3)\, \delta_L(1))$$

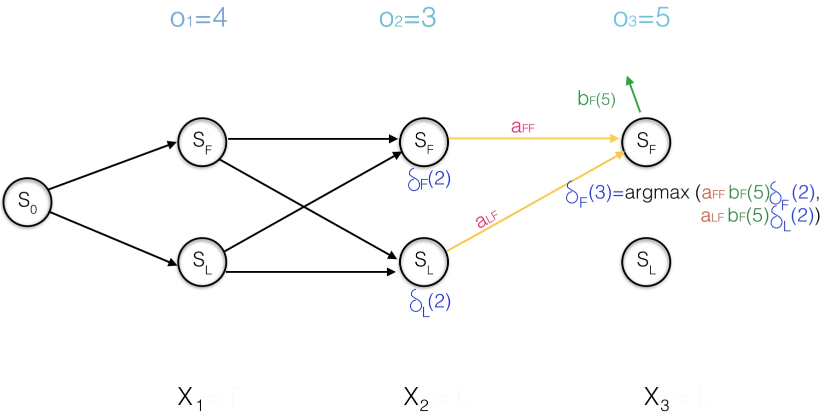
Viterbi algorithm, main step



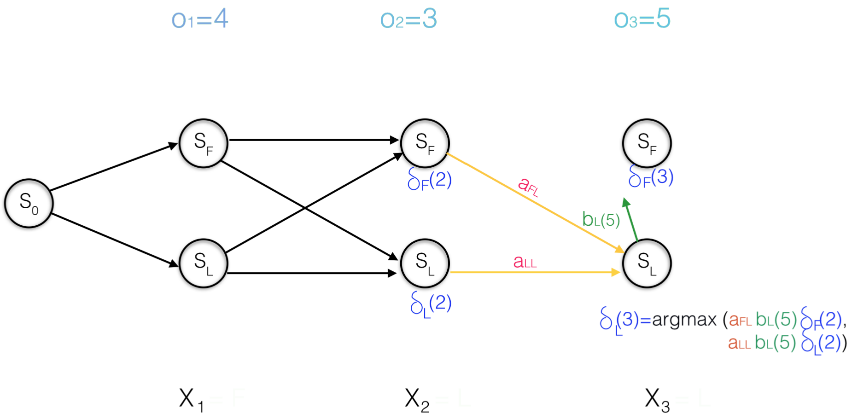
$\delta_L(2) = \text{argmax}\,(a_{FL}\, b_L(3)\, \delta_F(1),$
$a_{LL}\, b_L(3)\, \delta_L(1))$

$\Psi_L(2) = F$

Viterbi algorithm, main step, observation is 5



Viterbi algorithm, main step, observation is 5



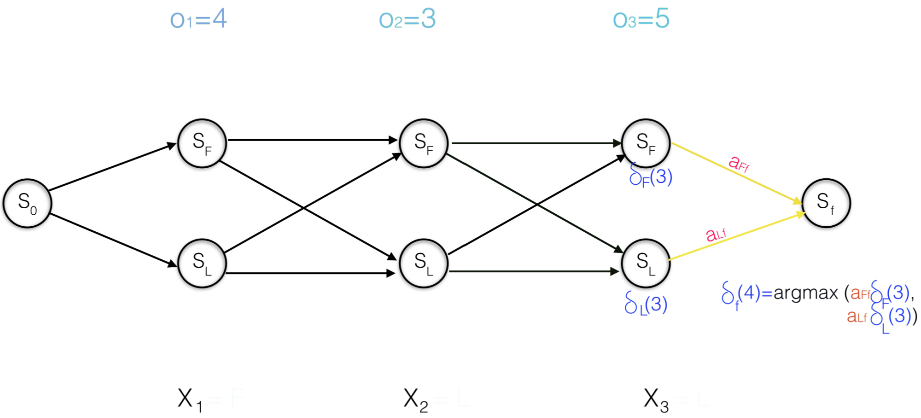
Viterbi algorithm, termination

- $\delta_f(T+1)$ is the probability of the entire state sequence up to point $T+1$ having been produced given the observation and the HMM's parameters.

$$P(X|O,\mu) = \delta_f(T+1) = \max_{1 \le i \le N} \delta_i \cdot (T) a_{if}$$

- It is calculated by maximising over the $\delta_i(T) \cdot a_{if}$, almost as per usual
- Not quite as per usual, because the final state s_f does not emit, so there is no $b_i(o_T)$ to consider.

Viterbi algorithm, termination



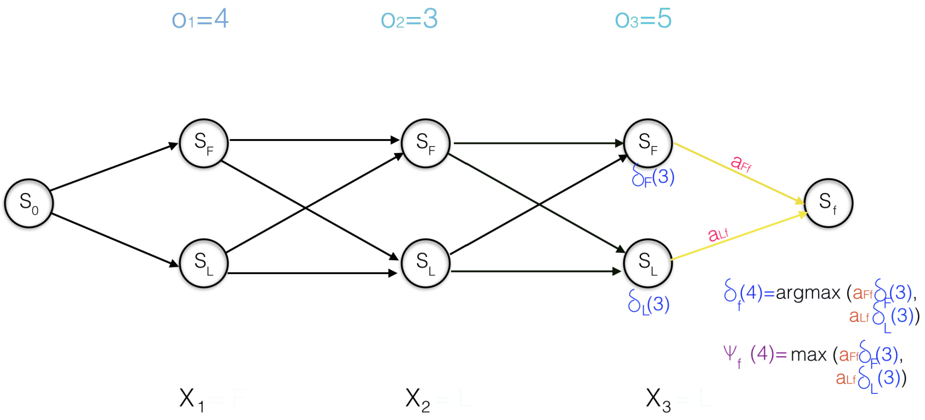
Viterbi algorithm, backtracing

- ψ_f is again calculated analogously to δ_f.

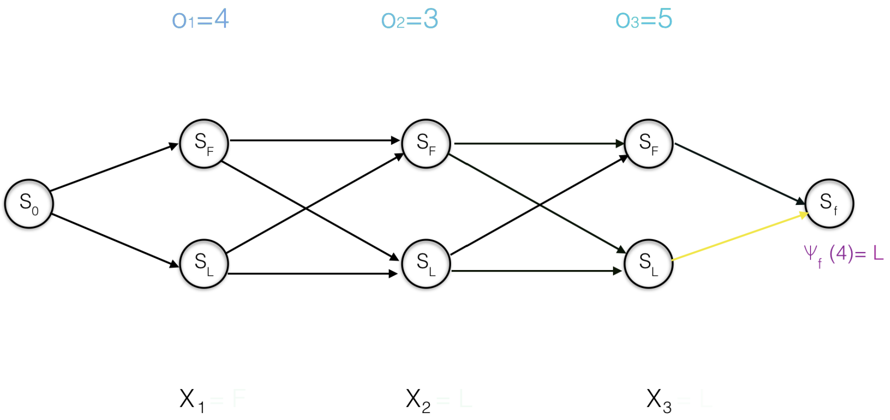
$$\psi_f(T+1) = \underset{1 \leq i \leq N}{\operatorname{argmax}} \, \delta_i(T) \cdot a_{if}$$

- It records X_T, the last state of the optimal state sequence.
- We will next go back to the cell concerned and look up **its** ψ to find the second-but-last state, and so on.

Viterbi algorithm, backtracing



Viterbi algorithm, backtracing



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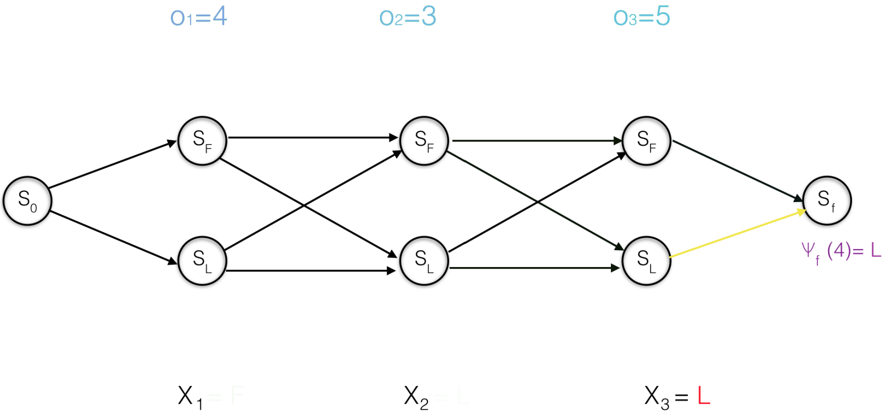
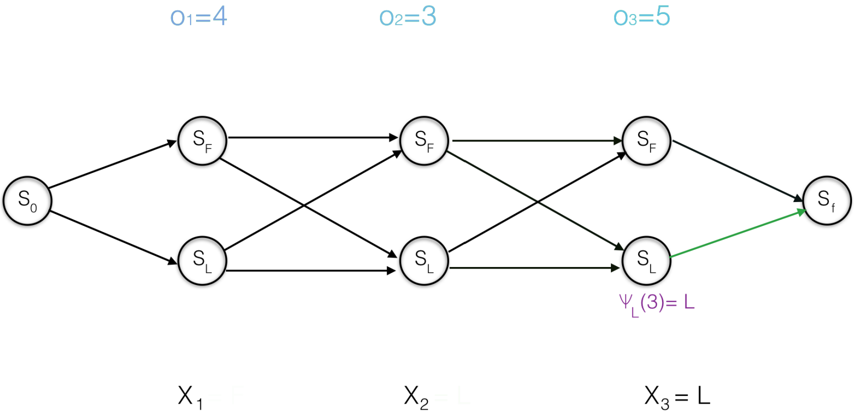
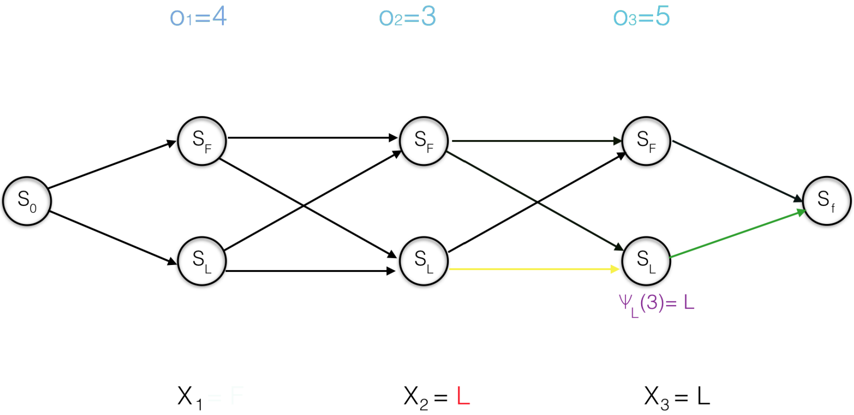


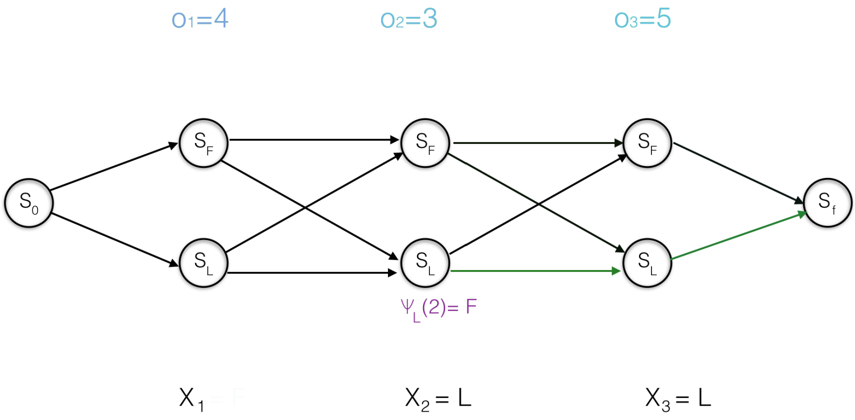
Viterbi algorithm, backtracing



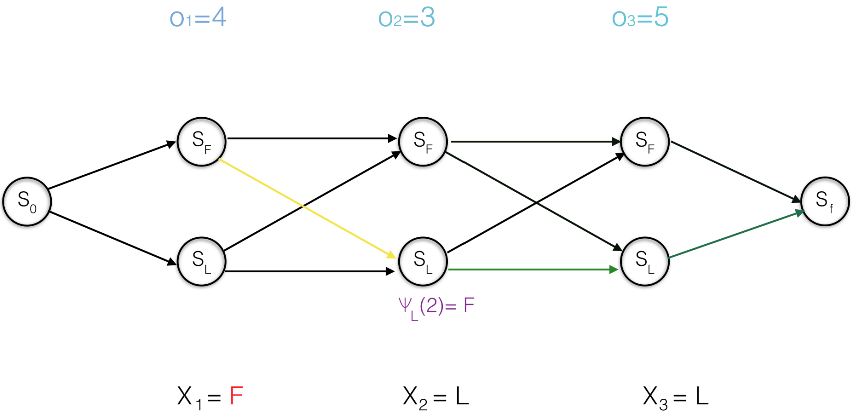
Viterbi algorithm, backtracing



Viterbi algorithm, backtracing



Viterbi algorithm, backtracing



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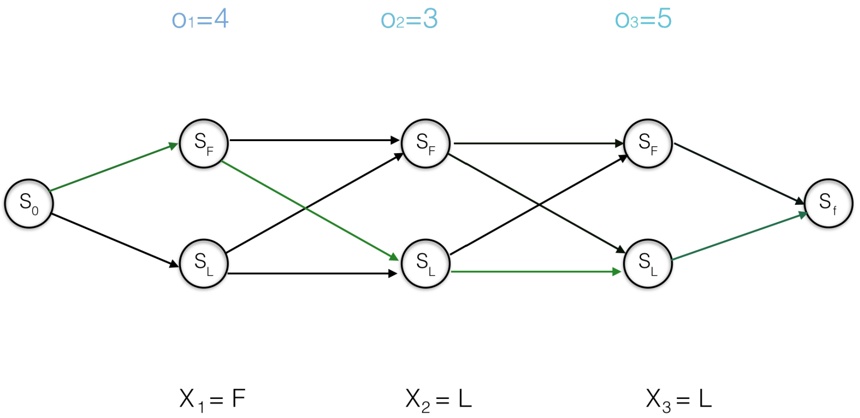

Precision and Recall

- So far we have measured system success in accuracy or agreement in Kappa.
- But sometimes it's only one type of example that we find interesting.
- We don't want a summary measure that averages over interesting and non-interesting examples, as accuracy does.
- In those cases we use precision, recall and F-measure.
- These metrics are imported from the field of information retrieval, where the difference beween interesting and non-interesting examples is particularly high.

Precision and Recall

System says:

Truth is:		F	L	Total
	F	a	b	a+b
	L	c	d	c+d
	Total	a+c	b+d	a+b+c+d

- Precision of L: $P_L = \frac{d}{b+d}$
- Recall of L: $R_L = \frac{d}{c+d}$
- F-measure of L: $F_L = \frac{2 P_L R_L}{P_L + R_L}$
- Accuracy: $A = \frac{a+d}{a+b+c+d}$

Your task today

Task 8:

- Implement the Viterbi algorithm.
- Run it on the dice dataset and measure precision of L (P_L), recall of L (R_L) and F-measure of L (F_L).

Ticking today

- Task 7 – HMM Parameter Estimation

Literature

- Manning and Schutze (2000). Foundations of Statistical Natural Language Processing, MIT Press. Chapter 9.3.2.
  - We use a state-emission HMM, but this textbook uses an arc-emission HMM. There is therefore a slight difference in the algorithm as to in which step the initial and final $b_j(k_t)$ are multiplied in.
- Jurafsky and Martin, 2nd Edition, chapter 6.4