Lambda calculus

(Advanced Functional Programming)

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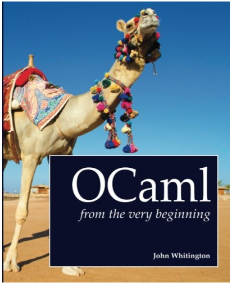
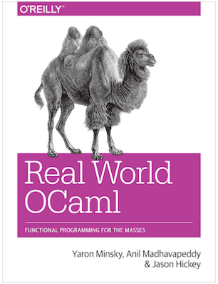
January 2017

Course outline

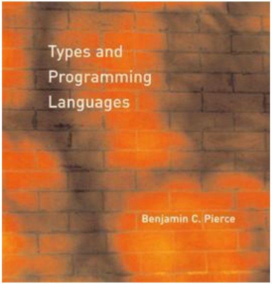
Books



OCaml from the very beginning
John Whitington
Coherent Press (2013)



Real World OCaml
Yaron Minsky,
Anil Madhavapeddy &
Jason Hickey
O'Reilly Media (2013)



Types and Programming Languages
Benjamin C. Pierce
MIT Press (2002)

Tooling

OPAM
OCaml package manager

Linux / OSX / VirtualBox

IOCaml

Fω

Fω interpreter

Philosophy and approach

- **practical**: with theory as necessary for understanding
- **real-world**: patterns and techniques from real applications
- **reusable**: general, broadly applicable techniques
- **current**: topics of ongoing research

Philosophy and approach

- **practical**: with theory as necessary for understanding
- **real-world**: patterns and techniques from real applications
- **reusable**: general, broadly applicable techniques
- **current**: topics of ongoing research
- **opinionated** (but you don't have to agree)

Mailing list

cl-acs-28@lists.cam.ac.uk

Announcements, questions and discussion. Feel free to post!

Have a question but feeling shy? Mail me directly and I'll anonymise and post your question:

jeremy.yallop@cl.cam.ac.uk

Exercises assessed and unassessed

Unassessed exercises

Useful preparation for assessed exercises; optional but recommended. Hand in for feedback, discuss freely on mailing list.

Assessed exercises

Mon 30 Jan	Mon 20 Feb	Mon 13 Mar
$\downarrow$	$\downarrow$	$\downarrow$
Mon 13 Feb	Mon 6 Mar	Tue 25 Apr
($\sim$30%)	($\sim$35%)	($\sim$35%)

Course structure

- **Technical background**
  Lambda calculus; type inference
- **Themes**
  Propositions as types; parametricity and abstraction
- **(Fancy) types**
  Higher-rank and higher-kinded polymorphism; modules and functors; generalised algebraic types; structured overloading
- **Patterns and techniques**
  Monads, applicatives, arrows, etc.; datatype-generic programming; staged programming
- **Applications**
  (E.g.) a foreign function library

Motivation & background

System Fω

Function composition in OCaml:

```
fun f g x -> f (g x)
```

Function composition in System Fω:

$$
\begin{array}{l}
\Lambda\alpha{::}*.\\
\quad\Lambda\beta{::}*.\\
\quad\quad\Lambda\gamma{::}*.\\
\quad\quad\quad\lambda \mathtt{f}{:}\alpha \to \beta.\\
\quad\quad\quad\quad\lambda \mathtt{g}{:}\gamma \to \alpha.\\
\quad\quad\quad\quad\quad\lambda \mathtt{x}{:}\gamma.\mathtt{f\ (g\ x)}
\end{array}
$$

What's the point of System Fω?

A framework for understanding language features and programming patterns:

- the elaboration language for type inference
- the proof system for reasoning with propositional logic
- the background for parametricity properties
- the language underlying higher-order polymorphism in OCaml
- the elaboration language for modules
- the core calculus for GADTs

Roadmap

$$
\begin{tikzcd}
 & F\omega \\
F \arrow[ur] & \\
\lambda^{\to} \arrow[u] &
\end{tikzcd}
$$

Inference rules

$$\frac{\begin{array}{c}\text{premise 1}\\ \text{premise 2}\\ \cdots\\ \text{premise N}\end{array}}{\text{conclusion}}\ \text{rule name}$$

Inference rules

$$\frac{\text{premise 1} \quad \text{premise 2} \quad \cdots \quad \text{premise N}}{\text{conclusion}}\ \text{rule name}$$

$$\frac{\text{all } M \text{ are } P \quad \text{all } S \text{ are } M}{\text{all } S \text{ are } P}\ \text{modus barbara}$$

Inference rules

$$\frac{\begin{array}{c}\text{premise 1}\\ \text{premise 2}\\ \cdots\\ \text{premise N}\end{array}}{\text{conclusion}}\ \text{rule name}$$

$$\frac{\begin{array}{c}\text{all } M \text{ are } P\\ \text{all } S \text{ are } M\end{array}}{\text{all } S \text{ are } P}\ \text{modus barbara}$$

$$\frac{\begin{array}{c}\text{all programs are buggy}\\ \text{all functional programs are programs}\end{array}}{\text{all functional programs are buggy}}\ \text{modus barbara}$$

Typing rules

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \to\text{-elim}$$

Terms, types, kinds

Kinds: K, K_1, K_2, ...

K is a kind

Environments: Γ

Γ is an environment

Types: A, B, C, ...

$\Gamma \vdash A :: K$

Terms: L, M, N, ...

$\Gamma \vdash M : A$

$$\lambda^{\rightarrow}$$

(simply typed lambda calculus)

$\lambda^{\rightarrow}$ by example

In $\lambda^{\rightarrow}$:

λx:A.x

```
λf:B → C.
  λg:A → B.
    λx:A.f (g x)
```

In OCaml:

```
fun x -> x
```

```
fun f g x -> f (g x)
```

Kinds in $\lambda^{\rightarrow}$

$$\frac{}{* \text{ is a kind}}\ *\text{-kind}$$

Kinding rules (type formation) in $\lambda^{\to}$

$$\frac{}{\Gamma \vdash \mathcal{B} :: *}\ \text{kind-}\mathcal{B}$$

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A \to B :: *}\ \text{kind-}\!\to$$

A kinding derivation

$$
\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathcal{B} :: *}\ \text{kind-}\mathcal{B} \qquad \dfrac{}{\Gamma \vdash \mathcal{B} :: *}\ \text{kind-}\mathcal{B}}{\Gamma \vdash \mathcal{B} \to \mathcal{B} :: *}\ \text{kind-}\to \qquad \dfrac{}{\Gamma \vdash \mathcal{B} :: *}\ \text{kind-}\mathcal{B}}{\Gamma \vdash (\mathcal{B} \to \mathcal{B}) \to \mathcal{B} :: *}\ \text{kind-}\to
$$

Environment formation rules

$$\frac{}{\cdot \text{ is an environment}}\ \Gamma\text{-}\cdot$$

$$\frac{\Gamma \text{ is an environment} \qquad \Gamma \vdash A :: *}{\Gamma, x : A \text{ is an environment}}\ \Gamma\text{-}:$$

Typing rules (term formation) in $\lambda^{\rightarrow}$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \text{tvar}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda \mathrm{x}{:}A.M : A \rightarrow B} \rightarrow\text{-intro} \qquad \frac{\Gamma \vdash M : A \rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \rightarrow\text{-elim}$$

A typing derivation for the identity function

$$\frac{\cdot, x : A \vdash x : A}{\cdot \vdash \lambda x{:}A.x : A \to A} \to\text{-intro}$$

Products by example

In $\lambda^{\rightarrow}$ with products:	**In OCaml**:
λp:(A $\rightarrow$ B) $\times$ A. fst p (snd p)	`fun (f,p) -> f p`
λx:A.$\langle$x,x$\rangle$	`fun x -> (x, x)`
λf:A $\rightarrow$ C. λg:B $\rightarrow$ C. λp:A $\times$ B. $\langle$f (fst p), g (snd p)$\rangle$	`fun f g (x,y) -> (f x, g y)`
λp.A $\times$ B.$\langle$snd p, fst p$\rangle$	`fun (x,y) -> (y,x)`

Kinding and typing rules for products

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A \times B :: *} \text{ kind-}\times$$

$$\frac{\begin{array}{c}\Gamma \vdash M : A \\ \Gamma \vdash N : B\end{array}}{\Gamma \vdash \langle M, N \rangle : A \times B} \times\text{-intro}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathbf{fst}\ M : A} \times\text{-elim-1}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathbf{snd}\ M : B} \times\text{-elim-2}$$

Sums by example

In $\lambda^{\rightarrow}$ with sums:

λf:A $\rightarrow$ C.
  λg:B $\rightarrow$ C.
    λs:A + B.
      case s **of**
        x.f x
      | y.g y

λs:A + B.
  case s **of**
    x.inr [B] x
  | y.inl [A] y

In OCaml:

```
fun f g s ->
  match s with
    Inl x -> f x
  | Inr y -> g y
```

```
function
  Inl x -> Inr x
| Inr y -> Inl y
```

Kinding and typing rules for sums

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A + B :: *}\ \text{kind-}+$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{inl}\ [B]\ M : A + B}\ +\text{-intro-1}$$

$$\frac{\Gamma \vdash N : B}{\Gamma \vdash \mathsf{inr}\ [A]\ N : A + B}\ +\text{-intro-2}$$

$$\frac{\begin{array}{c}\Gamma \vdash L : A + B \\ \Gamma, x : A \vdash M : C \\ \Gamma, y : B \vdash N : C\end{array}}{\Gamma \vdash \mathsf{case}\ L\ \mathsf{of}\ x.M \mid y.N : C}\ +\text{-elim}$$

System F

(polymorphic lambda calculus)

System F by example

$\Lambda\alpha{::}*.\lambda\mathtt{x}{:}\alpha.\mathtt{x}$

$$
\begin{array}{l}
\Lambda\alpha{::}*. \\
\quad\Lambda\beta{::}*. \\
\quad\quad\Lambda\gamma{::}*. \\
\quad\quad\quad\lambda\mathtt{f}{:}\beta \to \gamma. \\
\quad\quad\quad\quad\lambda\mathtt{g}{:}\alpha \to \beta. \\
\quad\quad\quad\quad\quad\lambda\mathtt{x}{:}\alpha.\mathtt{f}\ (\mathtt{g}\ \mathtt{x})
\end{array}
$$

$\Lambda\alpha{::}*.\Lambda\beta{::}*.\lambda\mathtt{p}{:}(\alpha \to \beta) \times \alpha.\ \mathbf{fst}\ \mathtt{p}\ (\mathbf{snd}\ \mathtt{p})$

New kinding rules for System F

$$\frac{\Gamma, \alpha{::}K \vdash A :: *}{\Gamma \vdash \forall \alpha{::}K.A :: *} \text{ kind-}\forall$$

$$\frac{\alpha{::}K \in \Gamma}{\Gamma \vdash \alpha :: K} \text{ tyvar}$$

New environment rule for System F

$$\frac{\Gamma \text{ is an environment} \qquad K \text{ is a kind}}{\Gamma, \alpha{::}K \text{ is an environment}}\ \Gamma\text{-}{::}$$

New typing rules for System F

$$\frac{\Gamma, \alpha{::}K \vdash M : A}{\Gamma \vdash \Lambda\alpha{::}K.M : \forall\alpha{::}K.A}\ \forall\text{-intro}$$

$$\frac{\Gamma \vdash M : \forall\alpha{::}K.A \qquad \Gamma \vdash B :: K}{\Gamma \vdash M\ [B] : A[\alpha{::=}B]}\ \forall\text{-elim}$$