L11: Algebraic Path Problems with applications to Internet Routing
Lectures 8 and 9

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Lecture 8

k shortest paths

- Recommended reading: Semiring frameworks and algorithms for shortest-distance problems, Mehryar Mohri, Journal of Automata, Languages and Combinatorics, v7, number 2, 2002

k shortest paths

The $\mathcal{T}_k$ semiring

$$\mathcal{T}_k \equiv (\mathbb{T}_k,\ \oplus_k,\ \otimes_k,\ \overline{0}_k,\ \overline{1}_k)$$

where

$$\begin{array}{rcl}
(a_0,\ \ldots,\ a_k) \oplus_k (b_0,\ \ldots,\ b_k) & \equiv & \min_k(a_0, \ldots,\ a_k,\ b_0,\ \ldots,\ b_k) \\
\overline{0}_k & \equiv & (\infty,\ \infty,\ \cdots,\ \infty) \\
(a_0,\ \ldots,\ a_k) \otimes_k (b_0,\ \ldots,\ b_k) & \equiv & \min_k(a_0 + b_0,\ a_0 + b_1,\ \ldots,\ a_k + b_k) \\
\overline{1}_k & \equiv & (0,\ \infty,\ \cdots,\ \infty)
\end{array}$$

$\mathcal{T}_k$ is $(k-1)$-stable.

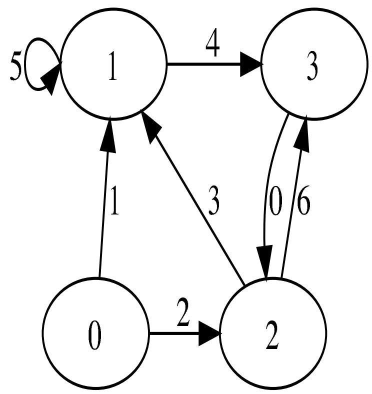
Examples ($\oplus_2$). Note that T_k is not idempotent for $k > 1$.

$$\begin{array}{rcl}
(5,\ 8) \oplus_2 (3,\ 6) & = & \min_2(5,\ 8,\ 3,\ 6) \\
 & = & (3,\ 5) \\
 & & \\
(1,\ 20) \oplus_2 (1,\ 20) & = & \min_2(1,\ 20,\ 1,\ 20) \\
 & = & (1,\ 1)
\end{array}$$

Examples ($\otimes_2$)

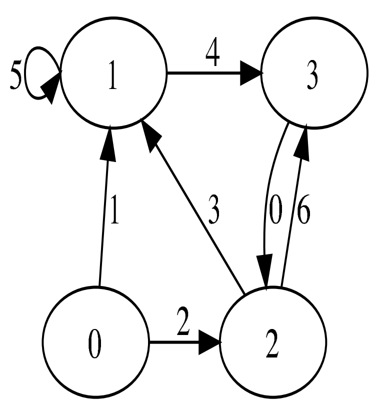
$$\begin{array}{rcl}
(5,\ 8) \otimes_2 (3,\ 6) & = & \min_2(5+3,\ 5+6,\ 8+3,\ 8+6) \\
 & = & \min_2(8,\ 11,\ 11,\ 14) \\
 & = & (8,\ 11) \\
 & & \\
(5,\ 8) \otimes_2 \overline{0}_2 & = & \min_2(5+\infty,\ 5+\infty,\ 8+\infty,\ 8+\infty) \\
 & = & \min_2(\infty,\ \infty,\ \infty,\ \infty) \\
 & = & (\infty,\ \infty) \\
 & = & \overline{0}_2
\end{array}$$

Mohri's example (here with $k = 3$)

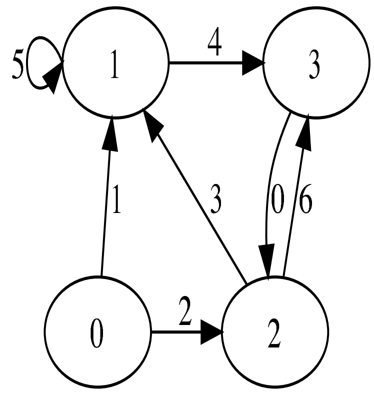


$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\mathbf{A} \;=\; \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc}
[\infty, \infty, \infty] & [1, \infty, \infty] & [2, \infty, \infty] & [\infty, \infty, \infty] \\
[\infty, \infty, \infty] & [5, \infty, \infty] & [\infty, \infty, \infty] & [4, \infty, \infty] \\
[\infty, \infty, \infty] & [3, \infty, \infty] & [\infty, \infty, \infty] & [6, \infty, \infty] \\
[\infty, \infty, \infty] & [\infty, \infty, \infty] & [0, \infty, \infty] & [\infty, \infty, \infty]
\end{array} \right]
\end{array}
$$

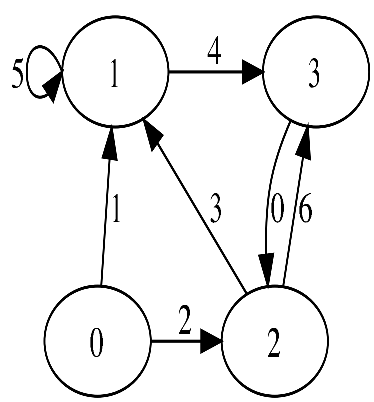
Red indicates change from previous iteration



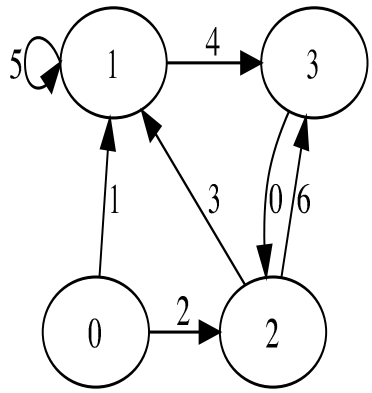
$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\
\mathbf{A}^{\langle 1 \rangle} \;=\; \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc}
[0, \infty, \infty] & [1, \infty, \infty] & [2, \infty, \infty] & [\infty, \infty, \infty] \\
[\infty, \infty, \infty] & [0, \textcolor{red}{5}, \infty] & [\infty, \infty, \infty] & [4, \infty, \infty] \\
[\infty, \infty, \infty] & [3, \infty, \infty] & [0, \infty, \infty] & [6, \infty, \infty] \\
[\infty, \infty, \infty] & [\infty, \infty, \infty] & [0, \infty, \infty] & [0, \infty, \infty]
\end{array} \right]
\end{array}
$$



$$\mathbf{A}^{\langle 2 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} [0, \infty, \infty] & [1, 5, 6] & [2, \infty, \infty] & [5, 8, \infty] \\ [\infty, \infty, \infty] & [0, 5, 10] & [4, \infty, \infty] & [4, 9, \infty] \\ [\infty, \infty, \infty] & [3, 8, \infty] & [0, 6, \infty] & [6, 7, \infty] \\ [\infty, \infty, \infty] & [3, \infty, \infty] & [0, \infty, \infty] & [0, 6, \infty] \end{bmatrix} \end{array}$$

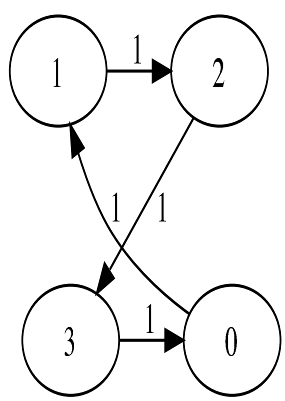


$$\mathbf{A}^{\langle 3 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} [0, \infty, \infty] & [1, 5, 6] & [2, 5, 8] & [5, 8, 9] \\ [\infty, \infty, \infty] & [0, 5, 7] & [4, 9, \infty] & [4, 9, 10] \\ [\infty, \infty, \infty] & [3, 8, 9] & [0, 6, 7] & [6, 7, 12] \\ [\infty, \infty, \infty] & [3, 8, \infty] & [0, 6, \infty] & [0, 6, 7] \end{bmatrix} \end{array}$$



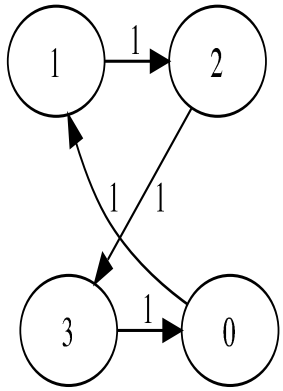
$$
\mathbf{A}^{\langle 4 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccc}
[0, \infty, \infty] & [1, 5, 6] & [2, 5, 8] & [5, 8, 9] \\
[\infty, \infty, \infty] & [0, 5, 7] & [4, 9, 10] & [4, 9, 10] \\
[\infty, \infty, \infty] & [3, 8, 9] & [0, 6, 7] & [6, 7, 12] \\
[\infty, \infty, \infty] & [3, 8, 9] & [0, 6, 7] & [0, 6, 7]
\end{array} \right] \end{array}
$$

Another example : a simple cycle.



$$
\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\left[\begin{array}{cccc}
[\infty, \infty, \infty] & [1, \infty, \infty] & [\infty, \infty, \infty] & [\infty, \infty, \infty] \\
[\infty, \infty, \infty] & [\infty, \infty, \infty] & [1, \infty, \infty] & [\infty, \infty, \infty] \\
[\infty, \infty, \infty] & [\infty, \infty, \infty] & [\infty, \infty, \infty] & [1, \infty, \infty] \\
[1, \infty, \infty] & [\infty, \infty, \infty] & [\infty, \infty, \infty] & [\infty, \infty, \infty]
\end{array} \right] \end{array}
$$

Solution $\mathbf{A}^*$ reached at 11-th iteration



$$\mathbf{A}^{\langle 11 \rangle} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccc} [0,4,8] & [1,5,9] & [2,6,10] & [3,7,11] \\ [3,7,11] & [0,4,8] & [1,5,9] & [2,6,10] \\ [2,6,10] & [3,7,11] & [0,4,8] & [1,5,9] \\ [1,5,9] & [2,6,10] & [3,7,11] & [0,4,8] \end{array}\right] \end{array}$$

Lecture 9

Marrtelli's semiring

- Recommended reading: A Gaussian Elimination Algorithm for the Enumeration of Cut Sets in a Graph. Alberto Martelli. Journal of the ACM (JACM). v23, number 1, 1976.

Reductions

If $(S, \oplus, \otimes)$ is a semiring and r is a function from S to S, then r is a reduction if for all a and b in S

1. $r(a) = r(r(a))$
2. $r(a \oplus b) = r(r(a) \oplus b) = r(a \oplus r(b))$
3. $r(a \otimes b) = r(r(a) \otimes b) = r(a \otimes r(b))$

Note that if either operation has an identity, then the first axioms is not needed. For example,

$$r(a) = r(a \oplus \overline{0}) = r(r(a) \oplus \overline{0}) = r(r(a))$$

Reduce operation

If $(S, \oplus, \otimes)$ is semiring and r is a reduction, then let $\mathrm{red}_r(S) = (S_r, \oplus_r, \otimes_r)$ where

1. $S_r = \{s \in S \mid r(s) = s\}$
2. $x \oplus_r y = r(x \oplus y)$
3. $x \otimes_r y = r(x \otimes y)$

Is the result always semiring?

Application of reduction

Let's try to build a semiring that uses **paths** to avoid counting to infinity!
First, a very useful construction:

$\mathrm{union\_lift}(S,\ \bullet)$

Assume $(S, \bullet)$ is a semigroup. Let

$$\mathrm{union\_lift}(S,\ \bullet) \equiv (\mathcal{P}_{\mathrm{fin}}(S),\ \cup,\ \hat{\bullet})$$

where

$$X \hat{\bullet} Y = \{x \bullet y \mid x \in X,\ y \in Y\},$$

and $X, Y \in \mathcal{P}_{\mathrm{fin}}(S)$, the set of finite subsets of S.

A semiring of elementary paths

Recall $\mathrm{paths}(E)$

$$\mathrm{paths}(E) \equiv \mathrm{union\_lift}(E^*,\ \cdot,\ \epsilon)$$

where $\cdot$ is sequence concatenation.

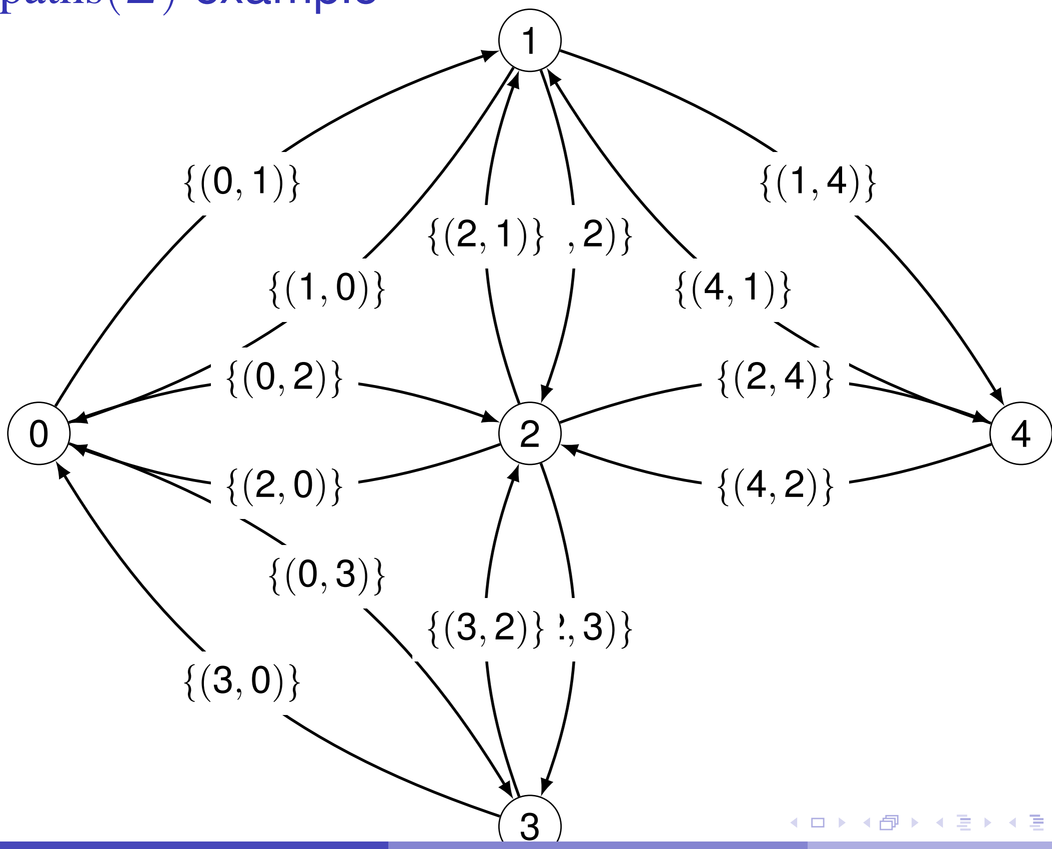
A path p is <u>elementary</u> if no node is repeated. Define the reduction

$$r(X) = \{p \in X \mid p \text{ is an elementary path}\}$$

Semiring of Elementary Paths

$$\mathrm{epaths}(E) = \mathrm{red}_r(\mathrm{paths}(E))$$

paths(E) example



paths(E) example, adjacency matrix

$$
\mathbf{I} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{ccccc}
\{\epsilon\} & \{\} & \{\} & \{\} & \{\} \\
\{\} & \{\epsilon\} & \{\} & \{\} & \{\} \\
\{\} & \{\} & \{\epsilon\} & \{\} & \{\} \\
\{\} & \{\} & \{\} & \{\epsilon\} & \{\} \\
\{\} & \{\} & \{\} & \{\} & \{\epsilon\}
\end{array}\right] \end{array}
$$

$$
\mathbf{A} \quad = \quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{ccccc}
\{\} & \{[(0,1)]\} & \{[(0,2)]\} & \{[(0,3)]\} & \{\} \\
\{[(1,0)]\} & \{\} & \{[(1,2)]\} & \{\} & \{[(1,4)]\} \\
\{[(2,0)]\} & \{[(2,1)]\} & \{\} & \{[(2,3)]\} & \{[(2,4)]\} \\
\{[(3,0)]\} & \{\} & \{[(3,2)]\} & \{\} & \{\} \\
\{\} & \{[(4,1)]\} & \{[(4,2)]\} & \{\} & \{\}
\end{array}\right] \end{array}
$$

Here I write a non-empty path p as $[p]$.

$\mathrm{paths}(E)$ example, solution

$$
\begin{array}{rcl}
\mathbf{A}^*(0,0) & = & \{\epsilon\} \\
\mathbf{A}^*(0,4) & = & \left\{ \begin{array}{l} [(0,1),(1,4)], \\ [(0,1),(1,2),(2,4)], \\ [(0,2),(2,4)], \\ [(0,2),(2,1),(1,4)], \\ [(0,3),(3,2),(2,4)], \\ [(0,3),(3,2),(2,1),(1,4)] \end{array} \right\}
\end{array}
$$

Now add some link weights ...

Let's use

$$\mathrm{AddZero}(\infty,\ (\mathbb{N},\ \mathsf{min},\ +)\ \vec{\times}\ \mathrm{paths}(E))$$

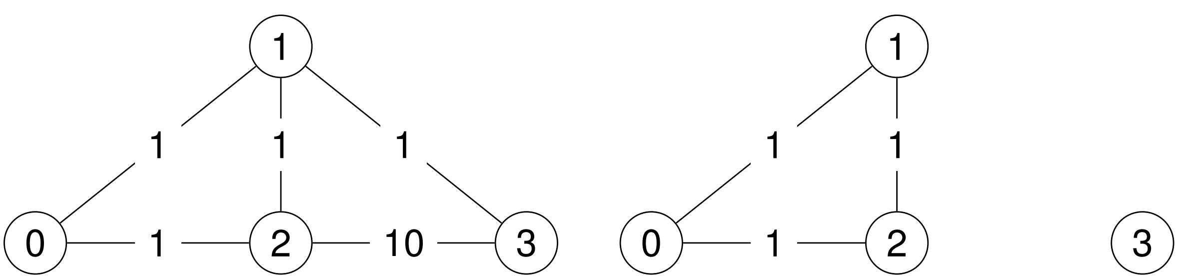
$$
\mathbf{I} \; = \;
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\
\left[\begin{array}{ccccc}
(0,\{\epsilon\}) & \infty & \infty & \infty & \infty \\
\infty & (0,\{\epsilon\}) & \infty & \infty & \infty \\
\infty & \infty & (0,\{\epsilon\}) & \infty & \infty \\
\infty & \infty & \infty & (0,\{\epsilon\}) & \infty \\
\infty & \infty & \infty & \infty & (0,\{\epsilon\})
\end{array} \right]
\end{array}
$$

$$
\mathbf{A} \; = \;
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \end{array} \\
\left[\begin{array}{ccccc}
\infty & (2,\{[(0,1)]\}) & (1,\{[(0,2)]\}) & (6,\{[(0,3)]\}) & \\
(2,\{[(1,0)]\}) & \infty & (5,\{[(1,2)]\}) & \infty & (4,\{[(\\
(1,\{[(2,0)]\}) & (5,\{[(2,1)]\}) & \infty & (4,\{[(2,3)]\}) & (3,\{[(\\
(6,\{[(3,0)]\}) & \infty & (4,\{[(3,2)]\}) & \infty & \\
\infty & (4,\{[(4,1)]\}) & (3,\{[(4,2)]\}) & \infty &
\end{array} \right.
\end{array}
$$

Solution

$$
\mathbf{A}^* = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c}
\begin{array}{ccc} \quad\quad 0 \quad\quad & \quad\quad 1 \quad\quad & \quad\quad 2 \quad\quad \end{array} \\
\begin{bmatrix}
(0, \{\epsilon\}) & (2, \{[(0,1)]\}) & (1, \{[(0,2)]\}) \\
(2, \{[(1,0)]\}) & (0, \{\epsilon\}) & (3, \{[(1,0),(0,2)]\}) \\
(1, \{[(2,0)]\}) & (3, \{[(2,0),(0,1)]\}) & (0, \{\epsilon\}) \\
(5, \{[(3,2),(2,0)]\}) & (7, \{[(3,2),(2,0),(0,1)]\}) & (4, \{[(3,2)]\}) \\
(4, \{[(4,2),(2,0)]\}) & (4, \{[(4,1)]\}) & (3, \{[(4,2)]\})
\end{bmatrix}
\end{array}
$$

Starting in an arbitrary state? No!



Let us try this again ...

Starting in an arbitrary state? No!

using

$$\mathrm{AddZero}(\infty,\ (\mathbb{N},\ \mathsf{min},\ +)\ \vec{\times}\ \mathrm{paths}(E))$$

$$
\begin{array}{c}
\vdots \quad \vdots \quad \vdots \\
\mathbf{B}_{998} \quad = \quad
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
(0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (999, \{\}) \\
(1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (999, \{\}) \\
(1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (999, \{\}) \\
\infty & \infty & \infty & (0, \{\epsilon\})
\end{array}\right]
\end{array} \\
\vdots \quad \vdots \quad \vdots
\end{array}
$$

Starting in an arbitrary state?

Solution: use another reduction!

$$
\begin{array}{rcl}
r(\mathrm{inr}(\infty)) & = & \mathrm{inr}(\infty) \\
r(\mathrm{inl})(s, W) & = & \begin{cases} \mathrm{inr}(\infty) & \text{if } W = \{\} \\ \mathrm{inl}(s, W) & \text{otherwise} \end{cases}
\end{array}
$$

Now use this instead

$$\mathrm{red}_r(\mathrm{AddZero}(\infty,\ (\mathbb{N},\ \mathsf{min},\ +)\ \vec{\times}\ \mathrm{paths}(E)))$$

Starting in an arbitrary state?

$\mathbf{B}_0$ and $\mathbf{B}_1$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3)]\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & (1, \{[(1,3)]\}) \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3)]\}) \\ (2, \{[(3,1),(1,0)]\}) & (1, \{[(3,1)]\}) & (2, \{[(3,1),(1,2)]\}) & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (2, \{[(0,1),(1,3)]\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (2, \{[(2,1),(1,3)]\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

Starting in an arbitrary state?

$\mathbf{B}_2$ and $\mathbf{B}_3$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & (3, \{[(0,2),(2,1),(1,3)]\}) \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & (3, \{[(2,0),(0,1),(1,3)]\}) \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} (0, \{\epsilon\}) & (1, \{[(0,1)]\}) & (1, \{[(0,2)]\}) & \infty \\ (1, \{[(1,0)]\}) & (0, \{\epsilon\}) & (1, \{[(1,2)]\}) & \infty \\ (1, \{[(2,0)]\}) & (1, \{[(2,1)]\}) & (0, \{\epsilon\}) & \infty \\ \infty & \infty & \infty & (0, \{\epsilon\}) \end{bmatrix} \end{array}$$

Min-set operations

Suppose $\leqslant$ is a partial order on S, $(S, \otimes)$ is a semigroup, and $X, Y \subseteq S$.

$$
\begin{array}{rcl}
\min_{\leqslant}(X) & \equiv & \{x \in X \mid \forall y \in X, \ \neg(y < x)\} \\
\mathcal{P}_{\mathrm{fin}}(S, \leqslant) & \equiv & \{X \subseteq S \mid X \text{ finite and } \min_{\leqslant}(X) = X\} \\
X \oplus^{\leqslant}_{\min} Y & \equiv & \min_{\leqslant}(X \cup Y) \\
X \otimes^{\leqslant}_{\min} Y & \equiv & \min_{\leqslant}(X \hat{\otimes} Y)
\end{array}
$$

Note that over $\mathcal{P}_{\mathrm{fin}}(S, \leqslant)$ the operation $\oplus^{\leqslant}_{\min}$ is always idempotent. However, $\otimes^{\leqslant}_{\min}$ may not be. Question: is $\min_{\leqslant}$ always a reduction?

Example over $\mathbb{N} \times \mathbb{N}$

$$(a, b) \preceq (c, d) \equiv (a \leqslant c) \wedge (b \leqslant d)$$

$$\min_{\preceq}(\{(10, 100), (9, 99), (99, 9), (99, 10)\})$$

$$= \{(9, 99), (99, 9)\}$$

$$\{(1, 0), (0, 1)\} \ (+ \times +)^{\leqslant}_{\min} \ \{(1, 0), (0, 1)\}$$

$$= \min_{\preceq}\left(\{(1, 0), (0, 1)\} \ (\widehat{+ \times +}) \ \{(1, 0), (0, 1)\}\right)$$

$$= \min_{\preceq}(\{(2, 0), (1, 1), (0, 2)\})$$

$$= \{(2, 0), (1, 1), (0, 2)\}$$

Observation

Incomparable relation

$$a \#_{\leqslant} b \equiv \neg(a \leqslant b) \wedge \neg(b \leqslant a)$$

Claim 12.1

If $x, y \in \min_{\leqslant}(X)$ and $x \neq y$, then $x \# y$.

Set like $\min_{\leqslant}(X)$ are often called (finite) antichains over $(S, \leqslant)$.

Suppose $D \in \{L, R\}$. Let $\leqslant \equiv \leqslant_{\oplus}^{D}$ and define

$$\begin{aligned}\mathcal{M}(D, (S, \oplus, \otimes)) &\equiv (\mathcal{P}_{\text{fin}}(S, \leqslant), \otimes_{\min}^{\leqslant}, \oplus_{\min}^{\leqslant}) \\ \mathcal{N}(D, (S, \oplus, \otimes)) &\equiv (\mathcal{P}_{\text{fin}}(S, \leqslant), \oplus_{\min}^{\leqslant}, \otimes_{\min}^{\leqslant})\end{aligned}$$

Recall: $a \leqslant_{\oplus}^{L} b \equiv a = a \oplus b$ and $a \leqslant_{\oplus}^{R} b \equiv b = a \oplus b$.

I suspect that 16 lectures could easily be dedicated to only $\mathcal{M}$ and $\mathcal{N}$ ...

IMPORTANT NOTE: So as not to reveal too much wrt Homework 2, I will assume for the rest of this lecture that $(S, \oplus, \otimes)$ has any conditions needed to guarantee that both $\mathcal{M}(D, (S, \oplus, \otimes))$ and $\mathcal{N}(D, (S, \oplus, \otimes))$ are semirings and that $\min_{\leqslant}$ acts as a reduction over $(\mathcal{P}(S), \cup, \hat{\otimes})$.

Looking at solutions over $\mathcal{N}$

$$(N,\ \boxplus,\ \boxtimes) \equiv \mathcal{N}(D,\ (S,\ \oplus,\ \otimes))$$

$$\begin{aligned}
\mathbf{A}^*(i,\ j) &= \bigboxplus_{p \in P(i,\ j)} \bigboxtimes_{e \in p} \mathbf{A}(e) \\
&= \min_{\leqslant}\Big(\bigcup_{p \in P(i,\ j)} \bigboxtimes_{e \in p} \mathbf{A}(e)\Big) \\
&= \min_{\leqslant}\Big(\bigcup_{p \in P(i,\ j)} \min_{\leqslant}\big(\hat{\bigotimes_{e \in p}} \mathbf{A}(e)\big)\Big) \\
&= \min_{\leqslant}\Big(\bigcup_{p \in P(i,\ j)} \hat{\bigotimes_{e \in p}} \mathbf{A}(e)\Big)
\end{aligned}$$

This assumes that $\min_{\leqslant}$ acts as a reduction.

What if $\oplus$ is selective?

Then

$$\min_{\leqslant}(X) \equiv \begin{cases} \{\} & (\text{if } X = \{\}) \\ \{x\} & (\text{if } X \neq \{\} \text{ and } x \text{ is } \leqslant\text{-least value in } X) \end{cases}$$

and for non-empty X and Y we have

$$X \boxtimes Y = \{x\} \boxtimes \{y\} = \min_{\leqslant}\{x \otimes y\} = \{x \otimes y\}.$$

$\mathrm{sp} \approx \mathcal{N}(L,\ (\mathbb{N},\ \min,\ +))$

$\mathrm{sp} \equiv \mathrm{AddZero}(\infty,\ (\mathbb{N},\ \min,\ +))$	$\mathcal{N}(L,\ (\mathbb{N},\ \min,\ +))$
$\mathrm{inl}(0)$	$\{0\}$
$\mathrm{inl}(17)$	$\{17\}$
$\mathrm{inr}(\infty)$	$\{\}$

So $\mathcal{N}$ is more interesting when $\oplus$ is not selective!

Let's compare

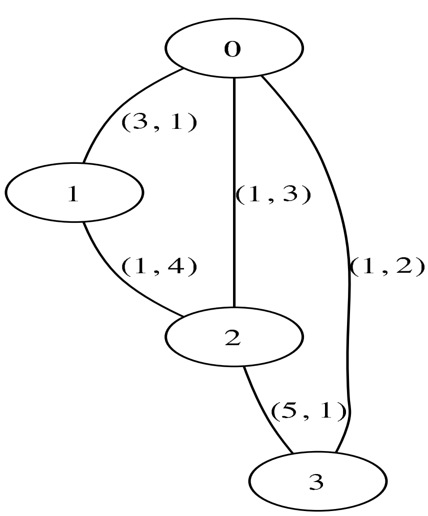
$$\mathrm{N}_1 \equiv \mathrm{AddZero}(\infty,\ (\mathbb{N},\ \mathsf{min},\ +) \times (\mathbb{N},\ \mathsf{min},\ +))$$

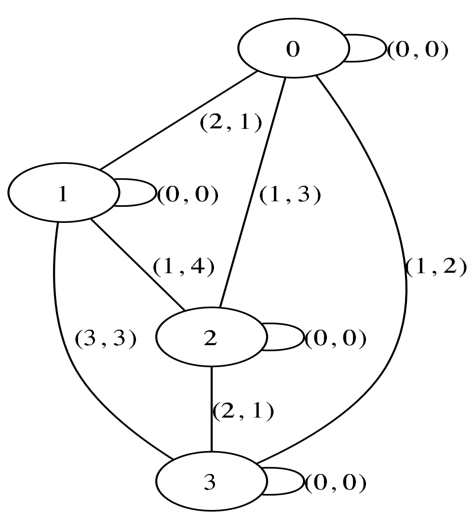
with

$$\mathrm{N}_2 \equiv \mathcal{N}(L,\ (\mathbb{N},\ \mathsf{min},\ +) \times (\mathbb{N},\ \mathsf{min},\ +))$$

Example with N_1

$\mathbf{A}_1$ $\mathbf{A}_1^*$

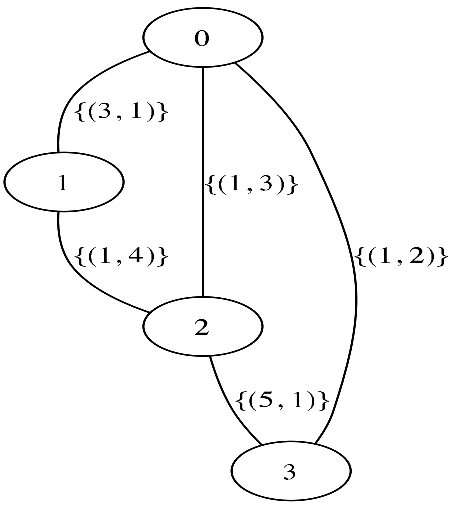




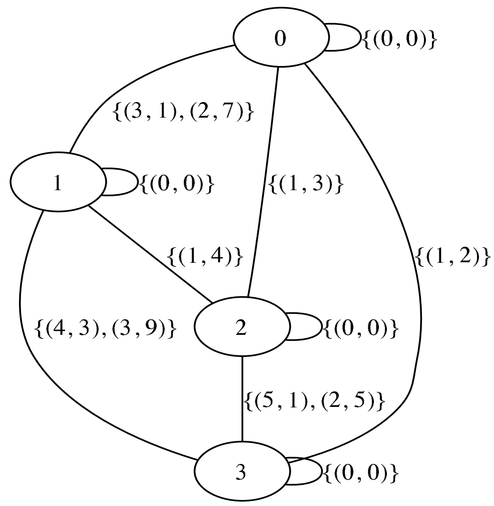
Each component is associated with a shortest path.

Example with N_2

$\mathbf{A}_2$



$\mathbf{A}_2^*$



Now add paths to N_2

Note that

$$((\mathbb{N},\ \min,\ +) \times (\mathbb{N},\ \min,\ +)) \vec{\times} \text{paths}(E)$$

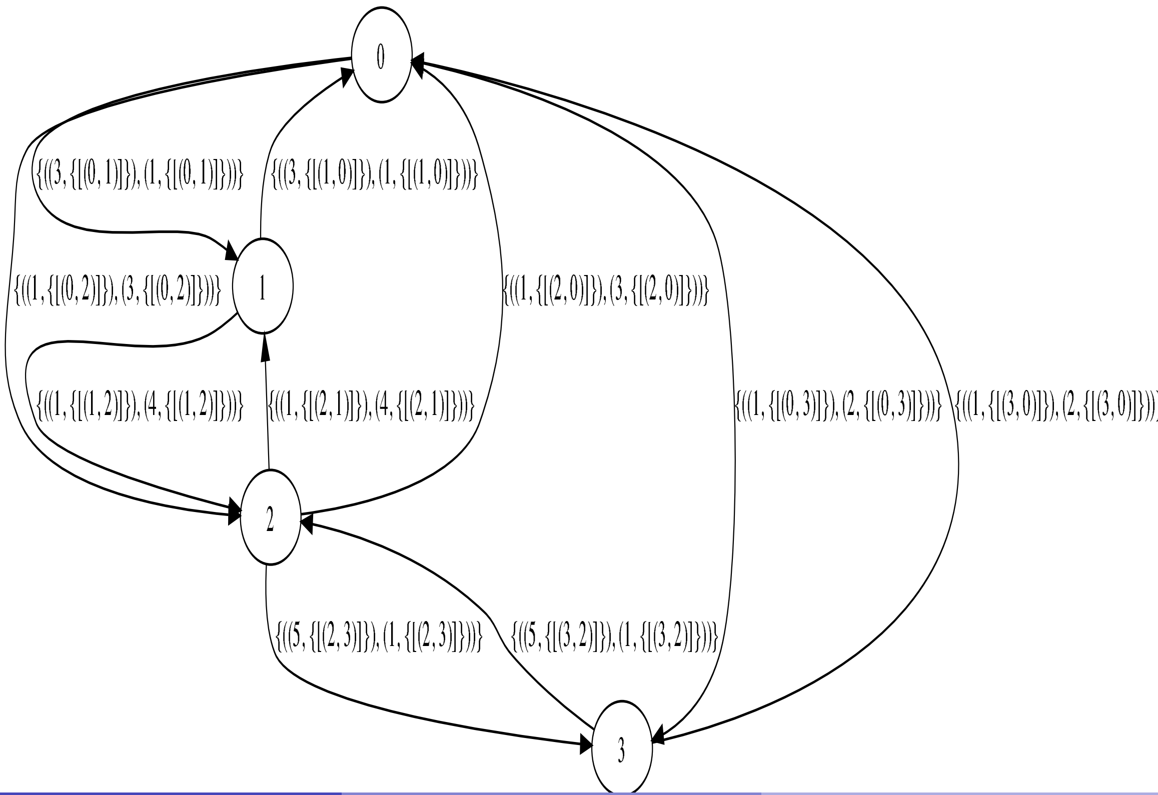
is not well-formed since $(\mathbb{N},\ \min,\ +) \times (\mathbb{N},\ \min,\ +)$ is not selective.

$$\mathrm{N}_3 \equiv \mathcal{N}(L,\ ((\mathbb{N},\ \min,\ +) \vec{\times} \text{paths}(E)) \times ((\mathbb{N},\ \min,\ +) \vec{\times} \text{paths}(E)))$$

N_3 has values of the form

$$\{((m_1, Q_1), (n_1, P_1)),\ ((m_2, Q_2), (n_2, P_2)),\ \cdots,\ ((m_k, Q_k), (n_k, P_k))\}$$

$\mathbf{A}_3$ over N_3



Compare!

$$\mathbf{A}_1^*(1,\ 3) \ = \ (3,\ 3)$$

$$\mathbf{A}_2^*(1,\ 3) \ = \ \{(4,\ 3),\ (3,\ 9)\}$$

$$\begin{aligned}\mathbf{A}_3^*(1,\ 3) \ = \ & \{((4, \{[(1,0), (0,3)]\}), (3, \{[(1,0), (0,3)]\})), \\ & ((3, \{[(1,2), (2,0), (0,3)]\}), (9, \{[(1,2), (2,0), (0,3)]\}))\}\end{aligned}$$

$$\mathbf{A}_1^*(0,\ 1) \ = \ (2, 1)$$

$$\mathbf{A}_2^*(0,\ 1) \ = \ \{(3, 1), (2, 7)\}$$

$$\begin{aligned}\mathbf{A}_3^*(0,\ 1) \ = \ & \{((3, \{[(0,1)]\}), (1, \{[(0,1)]\})), \\ & ((2, \{[(0,2), (2,1)]\}), (7, \{[(0,2), (2,1)]\}))\}\end{aligned}$$

Are we happy?

Hmmm, wait a minute!

$$\mathbf{A}_1^*(1,\ 2) \ = \ (1,\ 4)$$

$$\mathbf{A}_2^*(1,\ 2) \ = \ \{(1,\ 4)\}$$

$$\begin{aligned}\mathbf{A}_3^*(1,\ 2) \ = \ & \{((1, \{[(1,2)]\}), (4, \{[(1,2)]\})), \\ & ((9, \{[(1,0), (0,3), (3,2)]\}), (4, \{[(1,0), (0,3), (3,2)]\})), \\ & ((4, \{[(1,0), (0,2)]\}), (4, \{[(1,0), (0,2)]\}))\}\end{aligned}$$

What is going on? The order and the related notion of incomparability are both rather complicated ...

Now, turning to solutions over $\mathcal{M}$

$$(M,\ \boxplus,\ \boxtimes) \equiv \mathcal{M}(D,\ (S,\ \oplus,\ \otimes))$$

$$\begin{aligned}\mathbf{A}^*(i,\ j) &= \bigboxplus_{p \in P(i,\ j)} \bigboxtimes_{e \in p} \mathbf{A}(e) \\ &= \bigboxplus_{p \in P(i,\ j)} \min_{\leqslant} \left(\bigcup_{e \in p} \mathbf{A}(e) \right) \\ &= \min_{\leqslant} \left(\hat{\bigotimes}_{p \in P(i,\ j)} \min_{\leqslant} \left(\bigcup_{e \in p} \mathbf{A}(e) \right) \right) \\ &= \min_{\leqslant} \left(\hat{\bigotimes}_{p \in P(i,\ j)} \bigcup_{e \in p} \mathbf{A}(e) \right)\end{aligned}$$

This assumes that $\min_{\leqslant}$ acts as a reduction (hands waving ...)

Martelli's Semiring

Let $G = (V,\ E)$ be a directed graph.

$$\mathrm{M}_1(G) \equiv \mathcal{M}(R,\ (2^E,\ \cup,\ \cup))$$

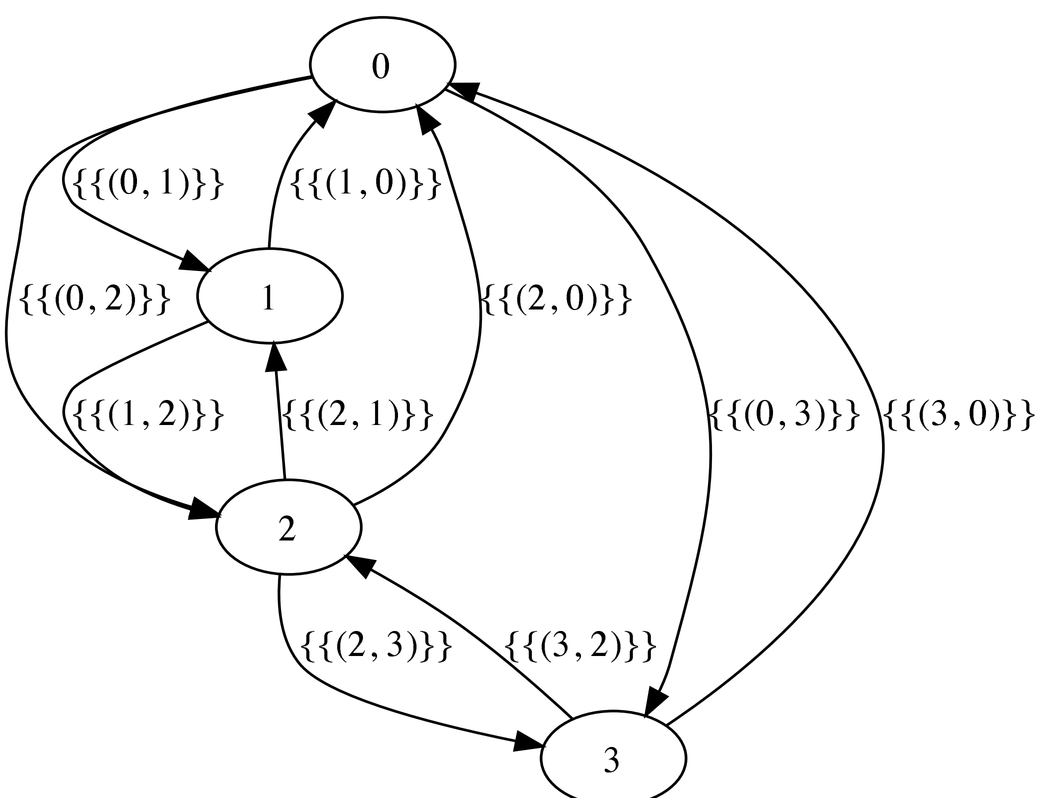
What does it do?

- If every arc $(i,\ j)$ is has weight $\mathbf{A}(i,\ j) = \{\{(i,\ j)\}\}$, then $\mathbf{A}^*(i,\ j)$ is the set of all minimal arc cut sets for i and j.

Definition

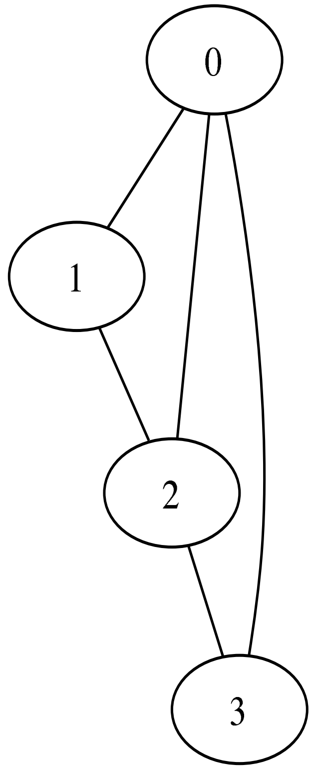
- A arc cut set $C \subseteq E$ for nodes i and j is a set of arcs such there is no path from i to j in the graph $(V,\ E - C)$.
- C is minimal if no proper subset of C is an arc cut set.

$\mathbf{A}_1$



Part of $\mathbf{A}_1^*$

$$
\begin{aligned}
\mathbf{A}_1^*(0,\ 1) &= \{\{(0,1),(2,1)\},\\
&\quad \{(0,1),(0,2),(0,3)\},\\
&\quad \{(0,1),(0,2),(3,2)\}\}\\
\mathbf{A}_1^*(0,\ 2) &= \{\{(0,2),(1,2),(3,2)\},\\
&\quad \{(0,1),(0,2),(3,2)\},\\
&\quad \{(0,1),(0,2),(0,3)\},\\
&\quad \{(0,2),(0,3),(1,2)\}\}\\
\mathbf{A}_1^*(2,\ 0) &= \{\{(2,0),(2,1),(3,0)\},\\
&\quad \{(1,0),(2,0),(3,0)\},\\
&\quad \{(1,0),(2,0),(2,3)\},\\
&\quad \{(2,0),(2,1),(2,3)\}\}\\
\mathbf{A}_1^*(2,\ 3) &= \{\{(2,0),(2,1),(2,3)\},\\
&\quad \{(0,3),(2,3)\},\\
&\quad \{(1,0),(2,0),(2,3)\}\}
\end{aligned}
$$



Homework 2

union_lift$(S,\ \bullet)$

Assume $(S, \bullet)$ is a semigroup. Let

$$\text{union\_lift}(S,\ \bullet) \equiv (\mathcal{P}_{\text{fin}}(S),\ \cup,\ \hat{\bullet})$$

where

$$X \hat{\bullet} Y = \{x \bullet y \mid x \in X,\ y \in Y\},$$

and $X, Y \in \mathcal{P}_{\text{fin}}(S)$, the set of finite subsets of S.

Let

$$\mathbb{SEMIRING}(S,\ \oplus,\ \otimes) \equiv (S,\ \oplus,\ \otimes) \text{ is a semiring}$$

Problem 1 (35 marks)

Find a $\mathbb{Q}_1$ such that

$$\mathbb{SEMIRING}(\mathcal{P}_{\text{fin}}(S),\ \cup,\ \hat{\bullet}) \Leftrightarrow \mathbb{Q}_1(S,\ \bullet).$$

Homework 2

Let $\leqslant$ be a partial order on S. For $X \subseteq S$, define

$$\min_{\leqslant}(X) \equiv \{x \in X \mid \forall y \in X,\ \neg(y < x)\}.$$

Define

$$\mathcal{P}_{\text{fin}}(S,\ \lesssim) \equiv \{X \subseteq S \mid X \text{ finite and } \min_{\leqslant}(X) = X\}.$$

Problem 2 (15 marks)

Prove that $(\mathcal{P}_{\text{fin}}(S,\ \leqslant),\ \oplus^{\leqslant}_{\text{min}})$ where

$$A \oplus^{\leqslant}_{\text{min}} B \equiv \min_{\leqslant}(A \cup B)$$

is a semigroup. It is clear that $\{\}$ is the identity. Is there always an annihilator?

Homework 2

Problem 3 (15 marks)

Let $(S,\ \otimes)$ be a semigroup over the ordered set $(S,\ \leqslant)$. Let

$$A \otimes^{\leqslant}_{\text{min}} B \equiv \min_{\leqslant}(\{a \otimes b \mid a \in A,\ b \in B\})$$

Find a $\mathbb{Q}_3$ such that

$$\mathbb{Q}_3(S,\ \leqslant,\ \otimes) \Leftrightarrow \mathbb{AS}(\mathcal{P}_{\text{fin}}(S,\ \leqslant),\ \otimes^{\leqslant}_{\text{min}}).$$

It is clear that $\{\}$ is the annihilator. Is there always an identity?

Homework 2

Suppose $S \equiv (S,\ \oplus,\ \otimes,\ \overline{0},\ \overline{1})$ is a semiring.

$$T \equiv (\mathcal{P}_{\mathrm{fin}}(S,\ \leqslant),\ \otimes^{\leqslant}_{\text{min}},\ \oplus^{\leqslant}_{\text{min}})$$

where $a \leqslant b \equiv a \oplus b = a$ (the left natural order).

Problem 4 (35 marks)

Find a $\mathbb{Q}_4$ such that

$$\mathbb{SEMIRING}(S) \Rightarrow (\mathbb{SEMIRING}(T) \Leftrightarrow \mathbb{Q}_4(S)).$$