L11: Algebraic Path Problems with applications to Internet Routing
Lectures 5 and 6

Timothy G. Griffin

timothy.griffin@cl.cam.ac.uk
Computer Laboratory
University of Cambridge, UK

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Semigroup properties (so far)

$$
\begin{array}{rcl}
\mathbb{AS}(S,\ \bullet) & \equiv & \forall a, b, c \in S,\ a \bullet (b \bullet c) = (a \bullet b) \bullet c \\
\mathbb{IID}(S,\ \bullet,\ \alpha) & \equiv & \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha \\
\mathbb{ID}(S,\ \bullet) & \equiv & \exists \alpha \in S,\ \mathbb{IID}(S,\ \bullet,\ \alpha) \\
\mathbb{IAN}(S,\ \bullet,\ \omega) & \equiv & \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega \\
\mathbb{AN}(S,\ \bullet) & \equiv & \exists \omega \in S,\ \mathbb{IAN}(S,\ \bullet,\ \omega) \\
\mathbb{CM}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b = b \bullet a \\
\mathbb{SL}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b \in \{a,\ b\} \\
\mathbb{IP}(S,\ \bullet) & \equiv & \forall a \in S,\ a \bullet a = a \\
\mathbb{IR}(S,\ \bullet) & \equiv & \forall s, t \in S, s \bullet t = t \\
\mathbb{IL}(S,\ \bullet) & \equiv & \forall s, t \in S, s \bullet t = s
\end{array}
$$

Recall that <u>is right</u> ($\mathbb{IR}$) and <u>is left</u> ($\mathbb{IL}$) are forced on us by wanting an $\Leftrightarrow$-rule for $\mathbb{SL}((S, \bullet) \times (T, \diamond))$

Bisemigroup properties (so far)

$$
\begin{array}{rcl}
\mathbb{AAS}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{AS}(S,\ \oplus) \\
\mathbb{AID}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{ID}(S,\ \oplus) \\
\mathbb{ACM}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{CM}(S,\ \oplus) \\
\mathbb{MAS}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{AS}(S,\ \otimes) \\
\mathbb{MID}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{ID}(S,\ \otimes) \\
\mathbb{LD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \\
\mathbb{RD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \\
\mathbb{ZA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{0} \in S,\ \mathbb{IID}(S,\ \oplus,\ \overline{0}) \wedge \mathbb{IAN}(S,\ \otimes,\ \overline{0}) \\
\hline
\mathbb{OA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{1} \in S,\ \mathbb{IID}(S,\ \otimes,\ \overline{1}) \wedge \mathbb{IAN}(S,\ \oplus,\ \overline{1}) \\
\mathbb{ASL}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{SL}(S,\ \oplus) \\
\mathbb{AIP}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{IP}(S,\ \oplus)
\end{array}
$$

Operations for adding a zero, a one

$$
\begin{array}{rcl}
\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes)) & \equiv & (S \uplus \{\overline{0}\},\ \oplus^{\mathrm{id}}_{\overline{0}},\ \otimes^{\mathrm{an}}_{\overline{0}}) \\
\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes)) & \equiv & (S \uplus \{\overline{1}\},\ \oplus^{\mathrm{an}}_{\overline{1}},\ \otimes^{\mathrm{id}}_{\overline{1}})
\end{array}
$$

Recall

$$
a \bullet^{\mathrm{id}}_{\alpha} b \equiv \begin{cases} a & (\text{if } b = \mathrm{inr}(\alpha)) \\ b & (\text{if } a = \mathrm{inr}(\alpha)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}
$$

$$
a \bullet^{\mathrm{an}}_{\omega} b \equiv \begin{cases} \mathrm{inr}(\omega) & (\text{if } b = \mathrm{inr}(\omega)) \\ \mathrm{inr}(\omega) & (\text{if } a = \mathrm{inr}(\omega)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}
$$

We can “inherit” semigroup rules

Examples

$$\begin{array}{rcl}
\mathbb{ACM}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \equiv & \mathbb{CM}(\mathrm{AddId}(\overline{0},\ (S,\ \oplus))) \\
 & \Leftrightarrow & \mathbb{CM}(S,\ \oplus)
\end{array}$$

$$\begin{array}{rcl}
\mathbb{MID}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \equiv & \mathbb{ID}(\mathrm{AddAn}(\overline{0},\ (S,\ \otimes))) \\
 & \Leftrightarrow & \mathbb{ID}(S,\ \otimes)
\end{array}$$

Property management for $\mathrm{AddZero}$

“Inherited” rules

$$\begin{array}{rcl}
\mathbb{AAS}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{AS}(S,\ \oplus) \\
\mathbb{AID}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{ACM}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{CM}(S,\ \oplus) \\
\mathbb{ASL}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{SL}(S,\ \oplus) \\
\mathbb{AIP}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{IP}(S,\ \oplus) \\
\mathbb{MAS}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{AS}(S,\ \otimes) \\
\mathbb{MID}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ID}(S,\ \otimes)
\end{array}$$

Easy Exercises

$$\begin{array}{rcl}
\mathbb{LD}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \\
\mathbb{RD}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \\
\mathbb{ZA}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{OA}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{OA}(S,\ \oplus,\ \otimes)
\end{array}$$

Easy Exercises?

Consider left distributivity ($\mathbb{LD}$)

a	b	c	$a \otimes_0^{\text{an}} (b \oplus_0^{\text{id}} c)$	$(a \otimes_0^{\text{an}} b) \oplus_0^{\text{id}} (a \otimes_0^{\text{an}} c)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inl}(a' \otimes (b' \oplus c'))$	$\text{inl}((a' \otimes b') \oplus (a' \otimes c'))$
$\text{inr}(\overline{0})$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$
$\text{inl}(a')$	$\text{inr}(\overline{0})$	$\text{inl}(c')$	$\text{inl}(a' \oplus c')$	$\text{inl}(a' \oplus c')$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inr}(\overline{0})$	$\text{inl}(a' \oplus b')$	$\text{inl}(a' \oplus b')$
$\text{inl}(a')$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$
$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$

However, adding a one is more complicated!

Consider left distributivity ($\mathbb{LD}$)

a	b	c	$a \otimes_1^{\text{id}} (b \oplus_1^{\text{an}} c)$	$(a \otimes_1^{\text{id}} b) \oplus_1^{\text{an}} (a \otimes_1^{\text{id}} c)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inl}(a' \otimes (b' \oplus c'))$	$\text{inl}((a' \otimes b') \oplus (a' \otimes c'))$
$\text{inr}(\overline{1})$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inl}(b' \oplus c')$	$\text{inl}(b' \oplus c')$
$\text{inl}(a')$	$\text{inr}(\overline{1})$	$\text{inl}(c')$	$\text{inl}(a')$	$\text{inl}((a' \oplus (a' \otimes c'))$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inr}(\overline{1})$	$\text{inl}(a')$	$\text{inl}((a' \otimes b') \oplus a')$
$\text{inl}(a')$	$\text{inr}(\overline{1})$	$\text{inr}(\overline{1})$	$\text{inl}(a')$	$\text{inl}(a' \oplus a')$
$\text{inr}(\overline{1})$	$\text{inr}(\overline{1})$	$\text{inr}(\overline{1})$	$\text{inr}(\overline{1})$	$\text{inr}(\overline{1})$

What is this?

$a = (a \otimes b) \oplus a$

Suppose $\oplus$ is idempotent and commutative and we let $a \leqslant b \equiv a = a \oplus b$. We know that

$$b \leqslant c \Rightarrow a \otimes b \leqslant a \otimes c$$

since $b = b \oplus c$ implies $a \otimes b = a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$. That is $\otimes$ is order preserving.
Now $a = (a \otimes b) \oplus a$ is telling us something else, that

$$a \leqslant a \otimes b.$$

That is, that multiplication is inflationary.

Absorption

$\mathbb{AB}$sorption properties (name is from lattice theory)

$$\begin{array}{lcl} \mathbb{RAB}(S, \oplus, \otimes) & \equiv & \forall a, b \in S,\ a = (a \otimes b) \oplus a = a \oplus (a \otimes b) \\ \mathbb{LAB}(S, \oplus, \otimes) & \equiv & \forall a, b \in S,\ a = (b \otimes a) \oplus a = a \oplus (b \otimes a) \end{array}$$

Observations

$$\begin{array}{rcl} \mathbb{RAB}(S, \oplus, \otimes) \wedge \mathbb{ID}(S, \oplus) & \Rightarrow & \mathbb{IP}(S, \otimes) \\ \mathbb{LAB}(S, \oplus, \otimes) \wedge \mathbb{ID}(S, \oplus) & \Rightarrow & \mathbb{IP}(S, \otimes) \\ \mathbb{LD}(S, \oplus, \otimes) \wedge \mathbb{OA}(S, \oplus, \otimes) & \Rightarrow & \mathbb{RAB}(S, \oplus, \otimes) \\ \mathbb{RD}(S, \oplus, \otimes) \wedge \mathbb{OA}(S, \oplus, \otimes) & \Rightarrow & \mathbb{LAB}(S, \oplus, \otimes) \end{array}$$

Rules for absorption? Consider $\mathbb{RAB}$

AddZero

a	b	$(a \otimes_{\bar{0}}^{\text{an}} b) \oplus_{\bar{0}}^{\text{id}} a$	$a \oplus_{\bar{0}}^{\text{id}} (a \otimes_{\bar{0}}^{\text{an}} b)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}((a' \otimes b') \oplus a)$	$\text{inl}(a' \oplus (a' \otimes b'))$
$\text{inr}(\bar{0})$	$\text{inl}(b')$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$
$\text{inl}(a')$	$\text{inr}(\bar{0})$	$\text{inl}(a')$	$\text{inl}(a')$
$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$	$\text{inr}(\bar{0})$

$$\begin{array}{lcl}
\mathbb{RAB}(\text{AddZero}(\bar{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \\
\mathbb{LAB}(\text{AddZero}(\bar{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes)
\end{array}$$

Rules for absorption? Consider $\mathbb{RAB}$

AddOne

a	b	$(a \otimes_{\bar{1}}^{\text{id}} b) \oplus_{\bar{1}}^{\text{an}} a$	$a \oplus_{\bar{1}}^{\text{an}} (a \otimes_{\bar{1}}^{\text{id}} b)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}((a' \otimes b') \oplus a)$	$\text{inl}(a' \oplus (a' \otimes b'))$
$\text{inr}(\bar{1})$	$\text{inl}(b')$	$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$
$\text{inl}(a')$	$\text{inr}(\bar{1})$	$\text{inl}(a')$	$\text{inl}(a' \oplus a')$
$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$	$\text{inr}(\bar{1})$

Property management for AddOne

"Inherited" rules

$$
\begin{array}{rcl}
\mathbb{AAS}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{AS}(S,\ \oplus) \\
\mathbb{AID}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ID}(S,\ \oplus) \\
\mathbb{ACM}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{CM}(S,\ \oplus) \\
\mathbb{ASL}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{SL}(S,\ \oplus) \\
\mathbb{AIP}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{IP}(S,\ \oplus) \\
\mathbb{MAS}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{AS}(S,\ \otimes) \\
\mathbb{MID}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE}
\end{array}
$$

Property management for AddOne

$$
\begin{array}{rcl}
\mathbb{LD}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \wedge \mathbb{RAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{RD}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \wedge \mathbb{LAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{ZA}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ZA}(S,\ \oplus,\ \otimes) \\
\mathbb{OA}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{RAB}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus) \\
\mathbb{LAB}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus)
\end{array}
$$

We have to start somewhere!

S	$\oplus$	$\otimes$	$\overline{0}$	$\overline{1}$	$\mathbb{LD}$	$\mathbb{RD}$	$\mathbb{ZA}$	$\mathbb{OA}$	$\mathbb{LAB}$	$\mathbb{RAB}$
$\mathbb{N}$	min	$+$		0	$\star$	$\star$		$\star$	$\star$	$\star$
$\mathbb{N}$	max	$+$	0	0	$\star$	$\star$			$\star$	$\star$
$\mathbb{N}$	max	min	0		$\star$	$\star$	$\star$		$\star$	$\star$
$\mathbb{N}$	min	max		0	$\star$	$\star$		$\star$	$\star$	$\star$

Introducing Minimax

$$\begin{aligned} \text{minimax} &\equiv \text{AddZero}(\infty, (\mathbb{N}, \text{min}, \text{max})) \\ &= (\mathbb{N} \uplus \{\infty\}, \text{min}^{\text{id}}_{\infty}, \text{max}^{\text{an}}_{\infty}) \end{aligned}$$

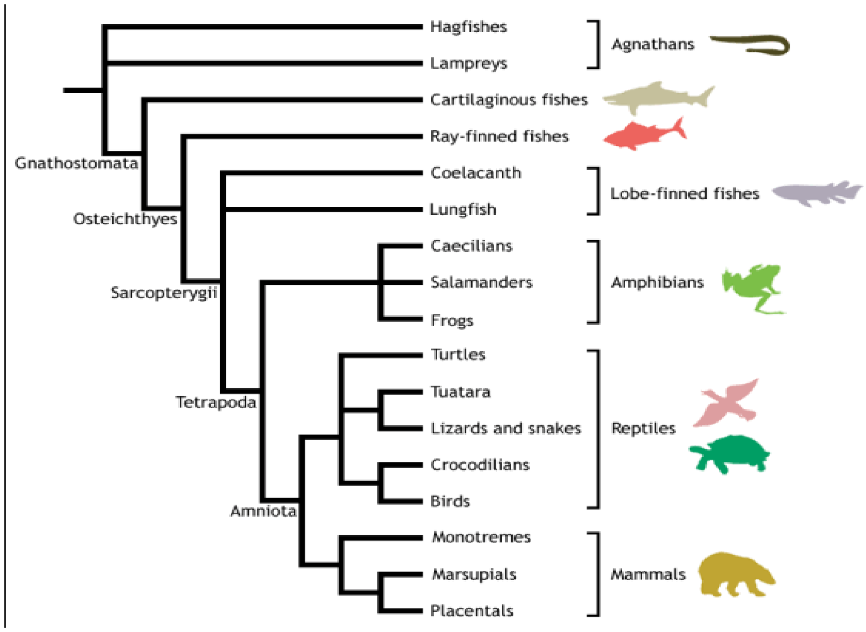
Some examples ...

$$\begin{aligned} \text{inl}(17)\ \text{min}^{\text{id}}_{\infty}\ \text{inr}(\infty) &= \text{inl}(17) \\ \text{inl}(17)\ \text{max}^{\text{an}}_{\infty}\ \text{inr}(\infty) &= \text{inr}(\infty) \end{aligned}$$

... which we will usually write as

$$\begin{aligned} 17 \text{ min } \infty &= 17 \\ 17 \text{ max } \infty &= \infty \end{aligned}$$

Dendrograms



http://www.instituteofcaninebiology.org/how-to-read-a-dendrogram.html

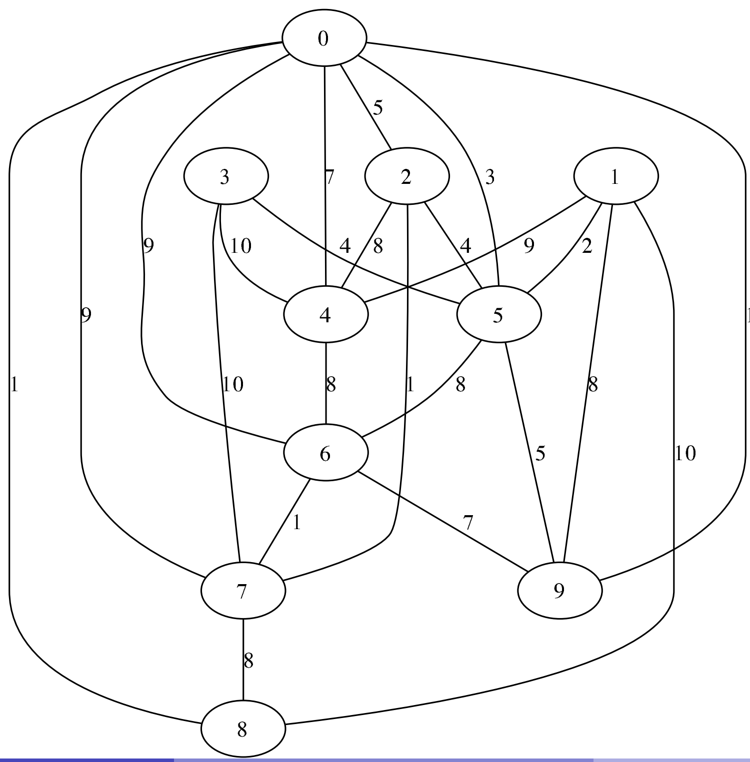
An application of Minimax

- Given an adjacency matrix **A** over minimax,
- suppose that $\mathbf{A}(i, j) = 0 \Leftrightarrow i = j$,
- suppose that **A** is symmetric ($\mathbf{A}(i, j) = \mathbf{A}(j, i)$),
- interpret $\mathbf{A}(i, j)$ as measured dissimilarity of i and j,
- interpret $\mathbf{A}^*(i, j)$ as inferred dissimilarity of i and j,

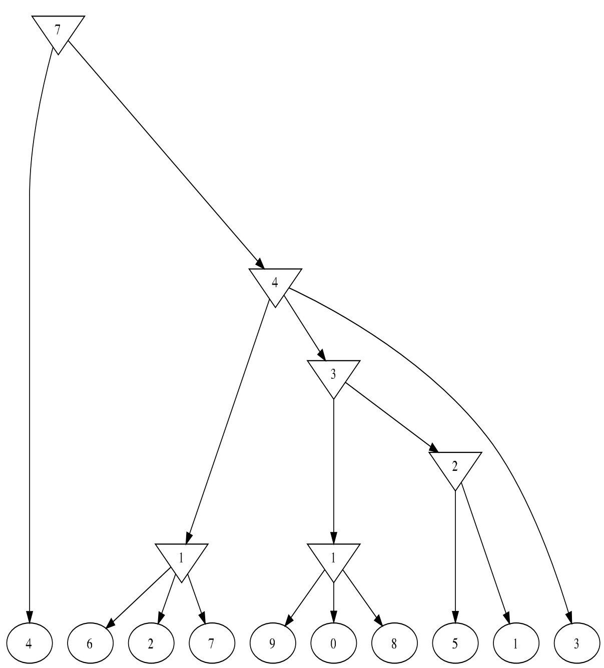
Many uses

- Hierarchical clustering of large data sets
- Classification in Machine Learning
- Computational phylogenetic
- ...

A (random) minimax matrix **A** drawn as a graph



The solution $\mathbf{A}^*$ drawn as a dendrogram



Hierarchical clustering? Why?

Suppose $(Y, \leqslant, +)$ is a totally ordered with least element 0.

Metric

A metric for set X over $(Y, \leqslant, +)$ is a function $d \in X \times X \to Y$ such that

- $\forall x, y \in X,\ d(x,\ y) = 0 \Leftrightarrow x = y$
- $\forall x, y \in X,\ d(x,\ y) = d(y,\ x)$
- $\forall x, y, z \in X,\ d(x,\ y) \leqslant d(x,\ z) + d(z,\ y)$

Ultrametric

An ultrametric for set X over $(Y, \leqslant)$ is a function $d \in X \times X \to Y$ such that

- $\forall x \in X,\ d(x,\ x) = 0$
- $\forall x, y \in X,\ d(x,\ y) = d(y,\ x)$
- $\forall x, y, z \in X,\ d(x,\ y) \leqslant d(x,\ z) \max d(z,\ y)$

Fun Facts

minimax and ultrametrics

If $\mathbf{A}$ is an $n \times n$ symmetric minimax adjacency matrix, then $\mathbf{A}^*$ is a finite ultrametric for $\{0,\ 1,\ \ldots,\ n-1\}$ over $(\mathbb{N}^{\infty},\ \leqslant))$.

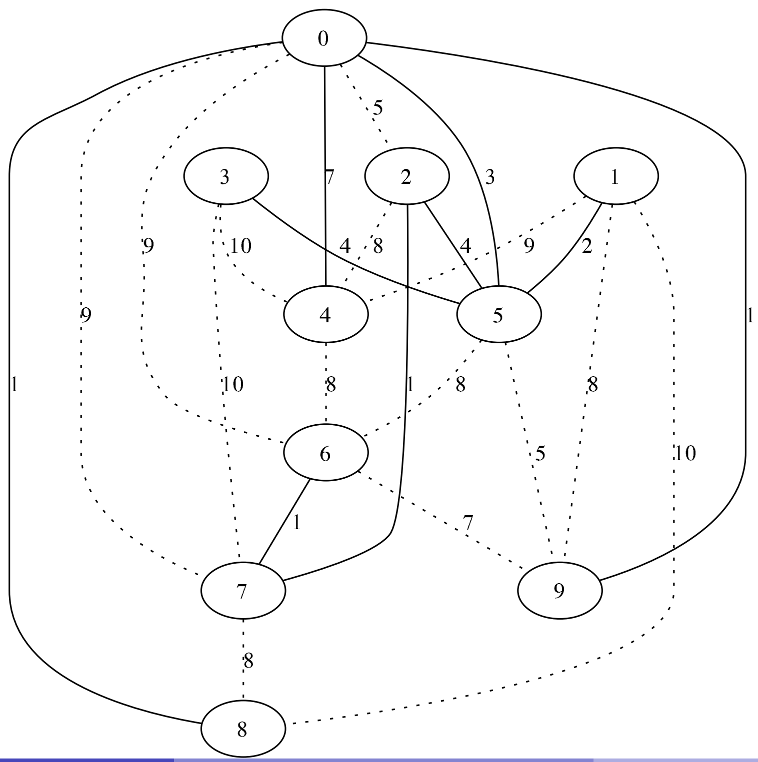
minimax and spanning trees

The set of arcs

$$\{(i,\ j) \in E \mid \mathbf{A}(i,\ j) = \mathbf{A}^*(i,\ j)\}$$

contain a spanning tree

A spanning tree derived from **A** and **A***



Recall

Lexicographic Product of Semigroups

Suppose that

$$\mathbb{AS}(S,\ \oplus_S) \wedge \mathbb{CM}(S,\ \oplus_S) \wedge \mathbb{SL}(S,\ \oplus_S) \wedge \mathbb{AS}(T,\ \oplus_T).$$

Let

$$(S,\ \oplus_S) \ \vec{\times}\ (T,\ \oplus_T) \equiv (S \times T,\ \oplus_S \vec{\times} \oplus_T)$$

where

$$(s_1,\ t_1)\ \oplus_S \vec{\times} \oplus_T\ (s_2,\ t_2) \equiv \begin{cases} (s_1 \oplus_S s_2,\ t_1 \oplus_T t_2) & s_1 = s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2,\ t_1) & s_1 = s_1 \oplus_S s_2 \neq s_2 \\ (s_1 \oplus_S s_2,\ t_2) & s_1 \neq s_1 \oplus_S s_2 = s_2 \end{cases}$$

Lexicographic product for Bi-semigroups

Suppose that

$$\mathbb{AS}(S,\ \oplus_S) \wedge \mathbb{CM}(S,\ \oplus_S) \wedge \mathbb{SL}(S,\ \oplus_S) \wedge \mathbb{AS}(T,\ \oplus_T).$$

Let

$$(S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T) \equiv (S \times T,\ \oplus_S \vec{\times} \oplus_T,\ \otimes_S \times \otimes_T)$$

Examples

$\oplus = \min \vec{\times} \max$, $\otimes = + \times \min$

$$\begin{array}{rcl} (3,10) \otimes ((17,21) \oplus (11,4)) & = & (3,10) \otimes (11,4) \\ & = & (14,4) \\ & & \\ ((3,10) \otimes (17,21)) \oplus ((3,10) \otimes (11,4)) & = & (20,10) \oplus (14,4) \\ & = & (14,4) \end{array}$$

$\oplus = \max \vec{\times} \min$, $\otimes = \min \times +$

$$\begin{array}{rcl} (3,10) \otimes ((17,21) \oplus (11,4)) & = & (3,10) \otimes (17,21) \\ & = & (3,31) \\ & & \\ ((3,10) \otimes (17,21)) \oplus ((3,10) \otimes (11,4)) & = & (3,31) \oplus (3,14) \\ & = & (3,14) \end{array}$$

Distributivity?

Theorem: If $\oplus_S$ is commutative and selective, then

$$\mathbb{LD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T)) \ \Leftrightarrow \\ \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T))$$

$$\mathbb{RD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T)) \ \Leftrightarrow \\ \mathbb{RD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{RD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{RC}(S,\ \otimes_S) \vee \mathbb{RK}(T,\ \otimes_T))$$

Left and Right Cancellative

$$\begin{array}{lcl} \mathbb{LC}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b \\ \mathbb{RC}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ a \bullet c = b \bullet c \Rightarrow a = b \end{array}$$

Left and Right Constant

$$\begin{array}{lcl} \mathbb{LK}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \\ \mathbb{RK}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ a \bullet c = b \bullet c \end{array}$$

Why bisemigroups?

But wait! How could any semiring satisfy either of these properties?

$$\begin{array}{lcl} \mathbb{LC}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b \\ \mathbb{LK}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \end{array}$$

- For $\mathbb{LC}$, note that we always have $\overline{0} \otimes a = \overline{0} \otimes b$, so $\mathbb{LC}$ could only hold when $S = \{\overline{0}\}$.
- For $\mathbb{LK}$, let $a = \overline{1}$ and $b = \overline{0}$ and $\mathbb{LK}$ leads to the conclusion that every c is equal to $\overline{0}$ (again!).

Normally we will add a zero and/or a one as the last step(s) of constructing a semiring. Alternatively, we might want to complicate our properties so that things work for semirings. A design trade-off!

Proof of $\Leftarrow$ for $\mathbb{LD}$

Assume

$$\begin{array}{ll} (1) & \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \\ (2) & \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \\ (3) & \mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T) \\ (4) & \mathbb{IP}(S,\ \oplus_S). \end{array}$$

Let $\oplus \equiv \oplus_S \vec{\times} \oplus_T$ and $\otimes \equiv \otimes_S \times \otimes_T$. Suppose

$$(s_1, t_1),\ (s_2, t_2),\ (s_3, t_3) \in S \times T.$$

We want to show that

$$\begin{array}{lll} \text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\ & = & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\ & \equiv & \text{rhs} \end{array}$$

Proof of $\Leftarrow$ for $\mathbb{LD}$

We have

$$\begin{array}{lll} \text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\ & = & (s_1,\ t_1) \otimes (s_2 \oplus_S s_3,\ t_{\text{lhs}}) \\ & = & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_1 \otimes_T t_{\text{lhs}}) \end{array}$$

$$\begin{array}{lll} \text{rhs} & \equiv & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\ & = & (s_1 \otimes_S s_2,\ t_1 \otimes_T t_2) \oplus (s_1 \otimes_S s_3,\ t_1 \otimes_T t_3) \\ & = & ((s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3),\ t_{\text{rhs}}) \\ & =_{(1)} & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_{\text{rhs}}) \end{array}$$

where t_{lhs} and t_{rhs} are determined by the appropriate case in the definition of $\oplus$. Finally, note that

$$\text{lhs} = \text{rhs} \Leftrightarrow t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}.$$

Proof by cases on $s_2 \oplus_S s_3$

Case 1 : $s_2 = s_2 \oplus_S s_3 = s_3$. Then $t_{\text{lhs}} = t_2 \oplus_T t_3$ and

$$t_1 \otimes_T t_{\text{lhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) =_{(2)} (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3).$$

Since $s_2 = s_3$ we have $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$ and

$$s_1 \otimes_S s_2 =_{(4)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3) =_{(4)} s_1 \otimes_S s_3.$$

Therefore,

$$t_{\text{rhs}} = (t_1 \otimes_T t_2) \oplus (t_1 \otimes_T t_3) = t_1 \otimes_T t_{\text{lhs}}.$$

Case 2 : $s_2 = s_2 \oplus_S s_3 \neq s_3$. Then $t_{\text{lhs}} = t_2$ and

$$t_1 \otimes_T t_{\text{lhs}} = t_1 \otimes_T t_2.$$

Since $s_2 = s_2 \oplus_S s_3$ we have

$$s_1 \otimes_S s_2 = s_1 \otimes_S (s_2 \oplus_S s_3) =_{(1)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3).$$

Case 2.1 $s_1 \otimes_S s_2 \neq s_1 \otimes_S s_3$. Then $t_{\text{rhs}} = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\text{lhs}}$.
Case 2.2 $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$. Then

$$t_{\text{rhs}} = (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) =_{(2)} t_1 \otimes_T (t_2 \oplus_T t_3)$$

We need to consider two subcases.
Case 2.2.1: Assume $\mathbb{LC}(S, \otimes_S)$. But $s_1 \otimes_S s_2 = s_1 \otimes_S s_3 \Rightarrow s_2 = s_3$, which is a contradiction.
Case 2.2.2 : Assume $\mathbb{LK}(T, \otimes_T)$. In this case we know

$$\forall a, b \in X,\ t_1 \otimes_T a = t_1 \otimes_T b.$$

Letting $a = t_2 \oplus_T t_3$ and $b = t_2$ we have

$$t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\text{lhs}}.$$

Case 3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to Case 2.

Other direction, $\Rightarrow$

Prove this:

$$\neg\mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \vee \neg\mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \vee (\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$$
$$\Rightarrow\ \neg\mathbb{LD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T)).$$

Case 1: $\neg\mathbb{LD}(S,\ \oplus_S,\ \otimes_S)$. That is

$$\exists a, b, c \in S,\ a \otimes_S (b \oplus_S c) \neq (a \otimes_S b) \oplus_S (a \otimes_S c).$$

Pick any $t \in T$. Then for some $t_1, t_2, t_3 \in T$ we have

$$\begin{array}{ll} & (a,\ t) \otimes ((b,\ t) \oplus (c,\ t)) \\ = & (a,\ t) \otimes (b \oplus_S c,\ t_1) \\ = & (a, \otimes_S (b \oplus_S c),\ t_2) \\ \neq & ((a \otimes_S b) \oplus_S (a \otimes_S c),\ t_3) \\ = & (a \otimes_S b,\ t \otimes_T t) \oplus (a \otimes_S c,\ t \otimes_T t) \\ = & ((a,\ t) \otimes (b,\ t)) \oplus ((a,\ t) \otimes (c,\ t)) \end{array}$$

Case 2: $\neg\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$. Similar.

Case 3: $(\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$. That is

$$\exists a, b, c \in S,\ c \otimes_S a = c \otimes_S b \wedge a \neq b$$

and

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

Since $\oplus_S$ is selective and $a \neq b$, we have $a = a \oplus_S b$ or $b = a \oplus_S b$.
Assume without loss of generality that $a = a \oplus_S b \neq b$.
Suppose that $t_1, t_2, t_3 \in T$. Then

$$\begin{array}{lll} \text{lhs} & \equiv & (c,\ t_1) \otimes ((a,\ t_2) \oplus (b,\ t_3)) \\ & = & (c,\ t_1) \otimes (a,\ t_2) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \end{array}$$

$$\begin{array}{lll} \text{rhs} & \equiv & ((c,\ t_1) \otimes (a,\ t_2)) \oplus ((c,\ t_1) \otimes (b,\ t_3)) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \oplus (c \otimes_S b,\ t_1 \otimes_T t_3) \\ & = & (c \otimes_S a,\ (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \end{array}$$

Our job now is to select t_1, t_2, t_3 so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) \equiv t_{\text{rhs}}.$$

We don't have very much to work with! Only

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

In addition, we can assume $\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$ (otherwise, use Case 2!), so

$$t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3).$$

We need to select t_1, t_2, t_3 so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq t_1 \otimes_T (t_2 \oplus_T t_3) \equiv t_{\text{rhs}}.$$

Case 3.1: $z \otimes_T x = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = y$, and $t_3 = x$ we have

$$t_{\text{lhs}} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.2: $z \otimes_T y = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T y = z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.3: $z \otimes_T x \neq z \otimes_T (x \oplus_T y) \neq z \otimes_T y$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

$\square$