L108 Assessment heads up

Assessed exercise sheet (ExSh#4)
(for 25% credit)

- issued Monday 7 Nov (in class)
- your answers are due by Monday 14 Nov, 16:00

(Take-home exam, 75% credit, in Jan.)

(Typed) Equations

$$\Gamma \vdash t = t' : A$$

(where $\Gamma \vdash t : A$ and $\Gamma \vdash t' : A$ hold) is satisfied by the semantics in a ccc if

$M[\![\Gamma \vdash t : A]\!]$ & $M[\![\Gamma \vdash t' : A]\!]$ are equal $\mathbb{C}$-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$

Q: which equations are always satisfied in any ccc?

A: $\beta\eta$-equivalence.

$\beta\eta$-Equality $\Gamma \vdash t =_{\beta\eta} t' : A$

where $\Gamma \vdash t : A$ & $\Gamma \vdash t' : A$,

is inductively defined, as follows:

- β-Conversions

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash s : A}{\Gamma \vdash (\lambda x : A . t) s =_{\beta\eta} t[s/x] : B}$$

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathrm{fst}(s, t) =_{\beta\eta} s : A} \qquad \frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathrm{snd}(s, t) =_{\beta\eta} t : B}$$

$\beta\eta$-Equality $\Gamma \vdash t =_{\beta\eta} t' : A$

where $\Gamma \vdash t : A$ & $\Gamma \vdash t' : A$,

is inductively defined, as follows:

- β-Conversions...
- η-Conversions

$$\frac{\Gamma \vdash t : A \to B \qquad x \# t}{\Gamma \vdash t =_{\beta\eta} (\lambda x : A . t\, x) : A \to B}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t =_{\beta\eta} (\mathrm{fst}\, t, \mathrm{snd}\, t) : A \times B} \qquad \frac{\Gamma \vdash t : \mathrm{unit}}{\Gamma \vdash t =_{\beta\eta} () : \mathrm{unit}}$$

$\beta\eta$-Equality $\Gamma \vdash t =_{\beta\eta} t' : A$

where $\Gamma \vdash t : A$ & $\Gamma \vdash t' : A$,

is inductively defined, as follows:

- β-Conversions...
- η-Conversions...
- Congruence rules

$$\frac{\Gamma, x : A \vdash t =_{\beta\eta} t' : B}{\Gamma \vdash \lambda x : A . t =_{\beta\eta} \lambda x : A . t' : A \to B}$$

$$\frac{\Gamma \vdash s =_{\beta\eta} s' : A \to B \qquad \Gamma \vdash t =_{\beta\eta} t' : A}{\Gamma \vdash s\, t =_{\beta\eta} s'\, t' : B}$$

etc.

$\beta\eta$-Equality $\Gamma \vdash t =_{\beta\eta} t' : A$

where $\Gamma \vdash t : A$ & $\Gamma \vdash t' : A$,

is inductively defined, as follows:

- β-conversions ...
- η-conversions ...
- Congruence rules ...
- $=_{\beta\eta}$ is reflexive, symmetric & transitive

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t =_{\beta\eta} t : A}$$

etc. (see p 9 of the notes)

$\beta\eta$-Equality $\Gamma \vdash t =_{\beta\eta} t' : A$

Soundness Theorem for ccc semantics of STLC

If $\Gamma \vdash t =_{\beta\eta} t' : A$, then in any ccc

$$M[\![\Gamma \vdash t : A]\!] = M[\![\Gamma \vdash t' : A]\!]$$

in $\mathbb{C}(M[\![\Gamma]\!], M[\![A]\!])$

(See Theorem 6.2 in the notes.)

E.g. given $\Gamma, x : A \vdash t : A'$ & $\Gamma \vdash t' : A$, then always have

$$M[\![\Gamma \vdash (\lambda x : A.\, t)\, t' : A']\!] = M[\![\Gamma \vdash t[t'/x] : A']\!]$$

(β-conversion is satisfied by the semantics in cccs)

E.g. given $\Gamma, x:A \vdash t : A'$ & $\Gamma \vdash t' : A$, then always have

$$M[\![\Gamma \vdash (\lambda x:A.t)t' : A']\!] = M[\![\Gamma \vdash t[t'/x] : A']\!]$$

(β-conversion is satisfied by the semantics in cccs)

because, if

$$M[\![\Gamma]\!] = X \qquad M[\![A]\!] = Y \qquad M[\![A']\!] = Z$$

$$M[\![\Gamma, x:A \vdash t : A']\!] = f : X \times Y \longrightarrow Z$$

$$M[\![\Gamma \vdash t' : A']\!] = g : X \longrightarrow Z$$

E.g. given $\Gamma, x : A \vdash t : A'$ & $\Gamma \vdash t' : A$, then always have

$$M[\![\Gamma \vdash (\lambda x : A . t)t' : A']\!] = M[\![\Gamma \vdash t[t'/x] : A']\!]$$

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$$M[\![\Gamma, x : A \vdash t : A']\!] = f : X \times Y \longrightarrow Z$$

$$M[\![\Gamma \vdash t' : A']\!] = g : X \longrightarrow Z$$

then

$$M[\![\Gamma \vdash \lambda x : A . t : A \to A']\!] = cur(f) : X \to Z^Y$$

$$M[\![\Gamma \vdash (\lambda x : A . t)t' : A']\!] = app \circ \langle cur(f), g \rangle$$

$M[\![\Gamma]\!] = X \quad M[\![A]\!] = Y \quad M[\![A']\!] = Z$

$M[\![\Gamma, x:A \vdash t : A']\!] = f : X \times Y \longrightarrow Z$

$M[\![\Gamma \vdash t' : A']\!] = g : X \longrightarrow Z$

$$M[\![\Gamma \vdash \lambda x{:}A.t : A \to A']\!] = cur(f) : X \longrightarrow Z^{Y}$$

$$\begin{aligned} M[\![\Gamma \vdash (\lambda x{:}A.t)t' : A']\!] &= app \circ \langle cur(f), g \rangle \\ &= app \circ (cur(f) \times id_Y) \circ \langle id_X, g \rangle \\ &= f \circ \langle id_X, g \rangle \\ &= M[\![\Gamma \vdash t[t'/x] : A']\!] \end{aligned}$$

by semantics of substitution...

Semantics of substitution in a ccc

<u>Theorem</u> If $\Gamma \vdash t' : A$ & $\Gamma, x : A \vdash t : A'$

then in any ccc

$$\begin{tikzcd}
M[\![\Gamma]\!] \arrow[rr, "{\langle id,\, M[\![\Gamma \vdash t' : A]\!]\rangle}"] \arrow[drr, "{M[\![\Gamma \vdash t[t'/x] : A']\!]}"'] & & M[\![\Gamma]\!] \times M[\![A]\!] \arrow[d, "{M[\![\Gamma, x : A \vdash t : A']\!]}"] \\
 & & M[\![A']\!]
\end{tikzcd}$$

commutes

(See Corollary 5.6 in the notes.)

The internal language of a ccc $\mathbb{C}$

- one ground type for each $\mathbb{C}$-object X
- one constant f^X for each $\mathbb{C}$-morphism $f : 1 \to X$ ("global element" of the object X)

Then types & terms of STLC over this language describe objects & morphisms of $\mathbb{C}$.

<u>For example</u> [Ex. Sh. 3, qu. 3], in any ccc $\mathbb{C}$ there is an isomorphism

$$\boxed{Z^{(X\times Y)} \cong (Z^Y)^X} \qquad (\text{any } X, Y, Z \in \text{obj}\,\mathbb{C})$$

which in the internal language of $\mathbb{C}$ is described by terms

$$\boxed{\begin{array}{ll} s & \Diamond \vdash \lambda f : (X\times Y) \to Z.\ \lambda x : X.\ \lambda y : Y.\ f(x,y) \\ & \quad : ((X\times Y) \to Z) \longrightarrow (X \to (Y \to Z)) \\ \hline t & \Diamond \vdash \lambda g : (X \to (Y \to Z)).\ \lambda z : X\times Y.\ g(\mathrm{fst}\, z)(\mathrm{snd}\, z) \\ & \quad : (X \to (Y \to Z)) \to ((X\times Y) \to Z) \end{array}}$$

satisfying
$$\begin{cases} \Diamond, f : (X\times Y) \to Z \vdash t(s\, f) =_{\beta\eta} f \\ \Diamond, g : X \to (Y \to Z) \vdash s(t\, g) =_{\beta\eta} g \end{cases}$$

Free cartesian closed categories

Soundness Theorem has a converse (completeness).

In fact, for a given set of ground types & typed constants, there's a single ccc $\boxed{\mathbb{F}}$ + interpretation function M so that

$\Gamma \vdash t =_{\beta\eta} t' : A$ holds iff the equation is satisfied by M in $\mathbb{F}$

Construction of $\mathbb{F}$

- objects of $\mathbb{F}$ are types of $\mathbb{T}$
- morphisms $A \to B$ in $\mathbb{F}$ are equivalence classes of closed terms $\diamond \vdash t : A \to B$ for equiv. relation

$$\boxed{t \sim t' \quad \text{if } \diamond \vdash t =_{\beta\eta} t' : A \to B}$$

Construction of $\mathbb{F}$

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$$\boxed{t \sim t' \text{ if } \diamond \vdash t =_{\beta\eta} t' : A \to B}$$

- identity on A is class of $\diamond \vdash \lambda x{:}A.\,x : A \to A$
- Composition induced by

$$\diamond \vdash t : A \to B,\ \diamond \vdash t' : B \to C \ \mapsto\ \diamond \vdash \lambda x{:}A.\,t'(t\,x) : A \to C$$

Curry-Howard

LOGIC		TYPE THEORY
propositions	$\leftrightarrow$	types
proofs	$\leftrightarrow$	terms

E.g. IPL vs STLC

proofs terms

Recall the derivation of $\varphi\Rightarrow\psi, \psi\Rightarrow\theta \vdash \varphi\Rightarrow\theta$ in IPL :

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{\cdots}\,(\mathrm{Ax})}{\Phi \vdash \psi\Rightarrow\theta}\,(\mathrm{Wk})
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{}{\cdots}\,(\mathrm{Ax})}{\cdots}\,(\mathrm{Wk})}{\Phi \vdash \varphi\Rightarrow\psi}\,(\mathrm{Wk})
\quad
\dfrac{}{\Phi \vdash \varphi}\,(\mathrm{Ax})
}{\Phi \vdash \psi}\,(\Rightarrow E)
}{\Phi \vdash \theta}\,(\Rightarrow E)
}{\varphi\Rightarrow\psi, \psi\Rightarrow\theta \vdash \varphi\Rightarrow\theta}\,(\Rightarrow I)
$$

$$\left(\Phi \triangleq \varphi\Rightarrow\psi, \psi\Rightarrow\theta, \varphi\right)$$

9:18

A corresponding STLC term:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(var)}
\UnaryInfC{$\cdots$}
\RightLabel{(var')}
\UnaryInfC{$\Phi \vdash z : \psi \Rightarrow \theta$}
\AxiomC{}
\RightLabel{(var)}
\UnaryInfC{$\cdots$}
\RightLabel{(var')}
\UnaryInfC{$\cdots$}
\RightLabel{(var')}
\UnaryInfC{$\Phi \vdash y : \varphi \Rightarrow \psi$}
\AxiomC{}
\RightLabel{(var)}
\UnaryInfC{$\Phi \vdash x : \varphi$}
\RightLabel{(app)}
\BinaryInfC{$\Phi \vdash yx : \psi$}
\RightLabel{(app)}
\BinaryInfC{$\Phi \vdash z(yx) : \theta$}
\RightLabel{(λ)}
\UnaryInfC{$y : \varphi \Rightarrow \psi,\ z : \psi \Rightarrow \theta \vdash \lambda x : \varphi .\ z(yx) : \varphi \Rightarrow \theta$}
\end{prooftree}
$$

$$\left(\Phi \triangleq [\, y : \varphi \Rightarrow \psi,\ z : \psi \Rightarrow \theta,\ x : \varphi \,] \right)$$

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Curry-Howard-Lawvere-Lambek

LOGIC		TYPE THEORY		CATEGORY TH.
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL vs STLC vs CCC's

proofs terms morphisms

Curry-Howard-Lawvere Lambek

these correspondences can be made into categorical <u>equivalences</u> - need the notions of <u>functor</u> & <u>natural transformation</u> to define "equivalence"...