Semantics of STLC terms in a ccc $\mathbb{C}$

Given a function M

constants $c^A \mapsto M(c^A) \in \mathbb{C}(1, M[\![A]\!])$

we get a function

provable typing $\Gamma \vdash t : A \mapsto M[\![\Gamma \vdash t : A]\!] \in \mathbb{C}(M[\![\Gamma]\!], M[\![A]\!])$

defined by recursing over the proof of $\Gamma \vdash t : A$ from the typing rules ...

Variables

$$\mathcal{M}[\![\Gamma, x : A \vdash x : A]\!] = \mathcal{M}[\![\Gamma]\!] \times \mathcal{M}[\![A]\!] \xrightarrow{\pi_2} \mathcal{M}[\![A]\!]$$

$$\mathcal{M}[\![\Gamma, x' : A' \vdash x : A]\!] = \mathcal{M}[\![\Gamma]\!] \times \mathcal{M}[\![A']\!]$$ (if $x' \notin \operatorname{dom} \Gamma$)

$$\begin{tikzcd} \mathcal{M}[\![\Gamma]\!] \times \mathcal{M}[\![A']\!] \arrow[d, "\pi_1"'] & \\ \mathcal{M}[\![\Gamma]\!] \arrow[r, "{\mathcal{M}[\![\Gamma \vdash x : A]\!]}"] & \mathcal{M}[\![A]\!] \end{tikzcd}$$

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\end{tikzcd}$$

(if $x' \notin \mathrm{dom}\,\Gamma$)

Constants

$$M[\![\Gamma \vdash c^A : A]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{M(c)} M[\![A]\!]$$

Variables

$$M[\![\Gamma, x : A \vdash x : A]\!] = M[\![\Gamma]\!] \times M[\![A]\!] \xrightarrow{\pi_2} M[\![A]\!]$$

$$M[\![\Gamma, x' : A' \vdash x : A]\!] = M[\![\Gamma]\!] \times M[\![A']\!] \qquad (\text{if } x' \notin \operatorname{dom}\Gamma)$$

$$\begin{array}{c} \downarrow \pi_1 \\ M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash x : A]\!]} M[\![A]\!] \end{array}$$

Constants

$$M[\![\Gamma \vdash c^A : A]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{M(c)} M[\![A]\!]$$

Unit value

$$M[\![\Gamma \vdash () : \mathrm{unit}]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1$$

Pairing

$$M[\![\Gamma \vdash (t,t') : A \times A']\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash t : A]\!],\, M[\![\Gamma \vdash t' : A']\!]\rangle} M[\![A]\!] \times M[\![A']\!]$$

Projections

$$M[\![\Gamma \vdash \mathrm{fst}\ t : A]\!] =$$

$$\begin{tikzcd} M[\![\Gamma]\!] \arrow[r, "{M[\![\Gamma \vdash t : A \times A']\!]}"] & M[\![A]\!] \times M[\![A']\!] \arrow[d, "\pi_1"] \\ & M[\![A]\!] \end{tikzcd}$$

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$$\boxed{\text{Given } \Gamma \vdash \mathrm{fst}\; t : A \text{ holds, there is a \underline{unique} } A' \text{ such that } \Gamma \vdash t : A \times A' \text{ holds}}$$

$$\boxed{\text{Lemma: if } \Gamma \vdash t : A \;\&\; \Gamma \vdash t : A',\ \text{then } A = A'}$$

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$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash t : A]\!],\, M[\![\Gamma \vdash t' : A']\!]\rangle} M[\![A]\!] \times M[\![A']\!]$$

Projections

$$M[\![\Gamma \vdash \mathrm{snd}\, t : A']\!] =$$

$$\begin{array}{l} M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t : A \times A']\!]} M[\![A]\!] \times M[\![A']\!] \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \pi_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad M[\![A']\!] \end{array}$$

$\boxed{\text{Given } \Gamma \vdash \mathrm{snd}\, t : A' \text{ holds, there is a unique } A \text{ such that } \Gamma \vdash t : A \times A' \text{ holds}}$

Function abstraction

$$\mathcal{M}[\![\Gamma \vdash \lambda x : A . t : A \to A']\!] =$$
$$\mathrm{cur}\left(\mathcal{M}[\![\Gamma]\!] \times \mathcal{M}[\![A]\!] \xrightarrow{\mathcal{M}[\![\Gamma, x : A \vdash t : A']\!]} \mathcal{M}[\![A']\!]\right)$$

Function abstraction

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Function application

$$M[\![\Gamma \vdash t\,t' : A']\!] =$$
$$M[\![\Gamma]\!] \xrightarrow{\langle f, f' \rangle} (M[\![A']\!])^{M[\![A]\!]} \times M[\![A]\!] \xrightarrow{app} M[\![A']\!]$$

where

A = unique type such that $\Gamma \vdash t : A \to A'$ holds (exists because $\Gamma \vdash t\,t' : A'$ holds)

$f = M[\![\Gamma \vdash t : A \to A']\!] : M[\![\Gamma]\!] \to (M[\![A']\!])^{M[\![A]\!]}$

$f' = M[\![\Gamma \vdash t' : A]\!] : M[\![\Gamma]\!] \to M[\![A]\!]$

Example

$$\left.\begin{array}{l} \Gamma = \emptyset,\ u : A \to B,\ v : B \to C \\ t = \lambda x : A.\ v(u\,x) \end{array}\right\}$$ so that $\Gamma \vdash t : A \to C$

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Suppose $M[\![A]\!] = X$, $M[\![B]\!] = Y$, $M[\![C]\!] = Z$ in $\mathbb{C}$

Then $M[\![\Gamma]\!] = (1 \times Y^X) \times Z^Y$

$M[\![\Gamma, x : A]\!] = ((1 \times Y^X) \times Z^Y) \times X$

$M[\![\Gamma, x : A \vdash v : B \to C]\!] = \pi_2 \circ \pi_1$

$M[\![\Gamma, x : A \vdash u : A \to B]\!] = \pi_2 \circ \pi_1 \circ \pi_1$

Example

$$\left.\begin{array}{l}\Gamma = \diamond, u : A \to B, v : B \to C \\ t = \lambda x : A.\ v(ux)\end{array}\right\}$$ so that $\Gamma \vdash t : A \to C$

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$M[\![\Gamma, x : A \vdash ux : B]\!] = app \circ \langle \pi_2 \circ \pi_1 \circ \pi_1, \pi_2 \rangle$

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Example

$\left.\begin{array}{l}\Gamma = \emptyset,\ u : A \to B,\ v : B \to C \\ t = \lambda x : A.\ v(ux)\end{array}\right\}$ so that $\Gamma \vdash t : A \to C$

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$M[\![\Gamma, x : A \vdash ux : B]\!] = app \circ \langle \pi_2 \circ \pi_1 \circ \pi_1, \pi_2 \rangle$

$M[\![\Gamma, x : A \vdash v(ux) : C]\!] = app \circ \langle \pi_2 \circ \pi_1, app \langle \pi_2 \circ \pi_1 \circ \pi_1, \pi_2 \rangle \rangle$

$M[\![\Gamma \vdash t : A \to C]\!] = cur(app \circ \cdots) : (1 \times Y^X) \times Z^Y \to Z^X$

(Typed) Equations

$$\Gamma \vdash t = t' : A$$

(where $\Gamma \vdash t : A$ and $\Gamma \vdash t' : A$ hold) is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash t : A]\!]$ & $M[\![\Gamma \vdash t' : A]\!]$ are equal $\mathbb{C}$-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$

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Q: which equations are always satisfied in any ccc?

A: $\beta\eta$-equivalence.

First need to define substitution & its semantics

Free variables $fv(t)$ of a term t

- $fv(c^A) = \emptyset$
- $fv(x) = \{x\}$
- $fv(()) = \emptyset$
- $fv((s,t)) = fv(s) \cup fv(t)$
- $fv(s\,t) = fv(s) \cup fv(t)$
- $fv(\lambda x : A.\, t) = fv(t) - \{x\}$

$\boxed{\{y \in fv(t) \mid y \neq x\}}$

Freshness relation:

$$x \# t \triangleq x \notin fv(t)$$

Substitution

$t'[t/x]$ = result of replacing all free occurrences of variable x in term t' by the term t, α-converting λ-bound variables in t' to avoid them "capturing" any free variables of t

E.g. $(\lambda y : A . (y, x))[y/x] \begin{cases} \text{is } \lambda z : A . (z, y) \\ \text{is } \underline{\text{NOT}}\ \lambda y : A . (y, y) \end{cases}$

Substitution

$$\frac{}{c^A[t/x] = c^A}$$

$$\frac{}{x[t/x] = t}$$

$$\frac{y \neq x}{y[t/x] = y}$$

$$\frac{}{()[t/x] = ()}$$

$$\frac{s_1[t/x] = s_1' \qquad s_2[t/x] = s_2'}{(s_1, s_2)[t/x] = (s_1', s_2')}$$

$$\frac{s[t/x] = s'}{(\mathrm{fst}\ s)[t/x] = \mathrm{fst}\ s'}$$

$$\frac{s[t/x] = s'}{(\mathrm{snd}\ s)[t/x] = \mathrm{snd}\ s'}$$

$$\frac{s[t/x] = s' \qquad y \# (x, t)}{(\lambda y : A.\, s)[t/x] = \lambda y : A.\, s'}$$

$$\frac{s_1[t/x] = s_1' \qquad s_2[t/x] = s_2'}{(s_1 s_2)[t/x] = s_1' s_2'}$$

Typing property of substitution

Substitution Lemma

If $\Gamma \vdash t : A$ & $\Gamma, x : A \vdash t' : A'$ (where $x \notin \Gamma$), then

$$\Gamma \vdash t'[t/x] : A'$$

(See Lemma 5.5 in the notes.)

Semantics of substitution in a ccc

Theorem If $\Gamma \vdash t : A$ & $\Gamma, x : A \vdash t' : A'$ then in any ccc

$$\begin{tikzcd}
M[\![\Gamma]\!] \arrow[rr, "{\langle id,\, M[\![\Gamma \vdash t : A]\!]\rangle}"] \arrow[drr, "{M[\![\Gamma \vdash t'[t/x] : A']\!]}"'] & & M[\![\Gamma]\!] \times M[\![A]\!] \arrow[d, "{M[\![\Gamma, x : A \vdash t' : A']\!]}"] \\
& & M[\![A']\!]
\end{tikzcd}$$

commutes

(See Corollary 5.6 in the notes.)