DIVERSION

Exercise Sheet 2, question 4

updated to make it more do-able

(explains what is a "monoid object" in a category with finite products)

If $\mathbb{C}$ is a category with a terminal object T and products $X \leftarrow X \times Y \rightarrow Y$ for all $X, Y \in \mathrm{obj}\,\mathbb{C}$,

then a monoid in $\mathbb{C}$ is given by

$M \in \mathrm{obj}\,\mathbb{C}$, $m \in \mathbb{C}(M \times M, M)$, $u : \mathbb{C}(T, M)$

such that these diagrams in $\mathbb{C}$

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such that these diagrams in $\mathbb{C}$

$$
\begin{tikzcd}
(M \times M) \times M \arrow[r, "m \times id"] \arrow[d, "\langle \pi_1\pi_1, \langle \pi_2\pi_1, \pi_2 \rangle\rangle"', "\cong"] & M \times M \arrow[r, "m"] & M \arrow[d, "id", "\cong"'] \\
M \times (M \times M) \arrow[r, "id \times m"] & M \times M \arrow[r, "m"] & M
\end{tikzcd}
$$

[c.f. $\forall x, y, z.\ m(m(x,y),z) = m(x, m(y,z))$]

If $\mathbb{C}$ is a category with a terminal object T and products $X \leftarrow X \times Y \rightarrow Y$ for all $X, Y \in \mathrm{obj}\,\mathbb{C}$,

then a **monoid in** $\mathbb{C}$ is given by

$M \in \mathrm{obj}\,\mathbb{C}$, $m \in \mathbb{C}(M \times M, M)$, $u : \mathbb{C}(T, M)$

such that these diagrams in $\mathbb{C}$

$$
\begin{tikzcd}
T \times M \arrow[r, "u \times id"] \arrow[d, "\pi_2"', "\cong"] & M \times M \arrow[r, "m"] & M \arrow[d, "id", "\cong"'] \\
M \arrow[rr, "id"'] & & M
\end{tikzcd}
$$

[Cf. $\forall x.\ m(u(*), x) = x$]

If $\mathbb{C}$ is a category with a terminal object T and products $X \leftarrow X \times Y \rightarrow Y$ for all $X, Y \in \mathrm{obj}\,\mathbb{C}$,

then a monoid in $\mathbb{C}$ is given by

$M \in \mathrm{obj}\,\mathbb{C}$, $m \in \mathbb{C}(M \times M, M)$, $u : \mathbb{C}(T, M)$

such that these diagrams in $\mathbb{C}$

$$\begin{tikzcd}
M \times T \arrow[r, "id \times u"] \arrow[d, "\pi_1"', "\cong"] & M \times M \arrow[r, "m"] & M \arrow[d, "id", "\cong"'] \\
M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

[c.f. $\forall x.\ m(x, u(*)) = x$]

END OF DIVERSION

NEXT UP

Simply Typed λ-Calculus

Intuitionistic Propositional Logic

$$\frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\,(\mathrm{Cut})$$

$$\frac{}{\Phi, \varphi \vdash \varphi}\,(\mathrm{Ax})$$

$$\frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\,(\mathrm{Wk})$$

$$\frac{}{\Phi \vdash \top}\,(\top)$$

$$\frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\,(\wedge I)$$

$$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\,(\Rightarrow I)$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\,(\wedge E_1)$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\,(\wedge E_2)$$

$$\frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\,(\Rightarrow E)$$

Recall the derivation of $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$:

$$
\dfrac{
\dfrac{
\dfrac{}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \psi \Rightarrow \theta}\,(\mathrm{Ax})
}{
\Phi \vdash \psi \Rightarrow \theta
}\,(\mathrm{Wk})
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\quad}\,(\mathrm{Ax})
}{\cdots}
}{\quad}\,(\mathrm{Wk})
}{\cdots}
}{\Phi \vdash \varphi \Rightarrow \psi}\,(\mathrm{Wk})
\qquad
\dfrac{}{\Phi \vdash \varphi}\,(\mathrm{Ax})
}{
\Phi \vdash \psi
}\,(\Rightarrow E)
}{
\dfrac{\Phi \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)
}\,(\Rightarrow E)
$$

$$\left(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi \right)$$

Another derivation of $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\cdots}\,(\text{Ax})
}{\cdots}\,(\text{Wk})
}{\Phi \vdash \varphi \Rightarrow \psi}\,(\text{Wk})
\qquad
\dfrac{}{\Phi \vdash \varphi}\,(\text{Ax})
}{\Phi \vdash \psi}\,(\Rightarrow E)
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\cdots}\,(\text{Ax})
}{\cdots}\,(\text{Wk})
}{\Phi, \psi \vdash \psi \Rightarrow \theta}\,(\text{Wk})
\qquad
\dfrac{}{\Phi, \psi \vdash \psi}\,(\text{Ax})
}{\Phi, \psi \vdash \theta}\,(\Rightarrow E)
}{
\dfrac{\Phi \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)
}\,(\text{Cut})
$$

$$\left(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi \right)$$

Proof Theory

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi\Rightarrow\psi,\psi\Rightarrow\theta\vdash\psi\Rightarrow\theta}(\mathrm{Ax})}{\Phi\vdash\psi\Rightarrow\theta}(\mathrm{Wk}) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}(\mathrm{Ax})}{\cdots}(\mathrm{Wk})}{\Phi\vdash\varphi\Rightarrow\psi}(\mathrm{Wk}) \quad \dfrac{}{\Phi\vdash\varphi}(\mathrm{Ax})}{\Phi\vdash\psi}(\Rightarrow E)}{\Phi\vdash\theta}(\Rightarrow E)}{\varphi\Rightarrow\psi,\psi\Rightarrow\theta\vdash\varphi\Rightarrow\theta}(\Rightarrow I)$$

$$(\Phi \triangleq \varphi\Rightarrow\psi,\psi\Rightarrow\theta,\varphi)$$

why is this proof simpler than this one ?

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}(\mathrm{Ax})}{\cdots}(\mathrm{Wk})}{\Phi\vdash\varphi\Rightarrow\psi}(\mathrm{Wk}) \quad \dfrac{}{\Phi\vdash\varphi}(\mathrm{Ax})}{\Phi\vdash\psi}(\Rightarrow E) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}(\mathrm{Ax})}{\cdots}(\mathrm{Wk})}{\Phi,\psi\vdash\psi\Rightarrow\theta}(\mathrm{Wk}) \quad \dfrac{}{\Phi,\psi\vdash\psi}(\mathrm{Ax})}{\Phi,\psi\vdash\theta}(\Rightarrow E)}{\Phi\vdash\theta}(\mathrm{Cut})}{\varphi\Rightarrow\psi,\psi\Rightarrow\theta\vdash\varphi\Rightarrow\theta}(\Rightarrow I)$$

$$(\Phi \triangleq \varphi\Rightarrow\psi,\psi\Rightarrow\theta,\varphi)$$

FACT: if $\Phi\vdash\varphi$ is derivable, it is derivable WITHOUT USING THE CUT RULE ("cut elimination")

Proof Theory

$$
\dfrac{
\dfrac{\dfrac{}{\varphi\Rightarrow\psi,\psi\Rightarrow\theta\vdash\psi\Rightarrow\theta}\,(\mathrm{Ax})}{\Phi\vdash\psi\Rightarrow\theta}\,(\mathrm{Wk})
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{\cdots}\,(\mathrm{Ax})}{\cdots}\,(\mathrm{Wk})}{\Phi\vdash\varphi\Rightarrow\psi}\,(\mathrm{Wk}) \quad \dfrac{}{\Phi\vdash\varphi}\,(\mathrm{Ax})}{\Phi\vdash\psi}\,(\Rightarrow\mathrm{E})
}{
\dfrac{\Phi\vdash\theta}{\varphi\Rightarrow\psi,\psi\Rightarrow\theta\vdash\varphi\Rightarrow\theta}\,(\Rightarrow\mathrm{I})
}\,(\Rightarrow\mathrm{E})
$$

$(\Phi \triangleq \varphi\Rightarrow\psi, \psi\Rightarrow\theta, \varphi)$

why is this proof simpler than this one ?

Need a language & calculus of proofs [for IPL] to answer questions like this...

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\dfrac{}{\cdots}\,(\mathrm{Ax})}{\cdots}\,(\mathrm{Wk})}{\Phi\vdash\varphi\Rightarrow\psi}\,(\mathrm{Wk}) \quad \dfrac{}{\Phi\vdash\varphi}\,(\mathrm{Ax})
}{\Phi\vdash\psi}\,(\Rightarrow\mathrm{E})
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{}{\cdots}\,(\mathrm{Ax})}{\cdots}\,(\mathrm{Wk})}{\Phi,\psi\vdash\psi\Rightarrow\theta}\,(\mathrm{Wk}) \quad \dfrac{}{\Phi,\psi\vdash\psi}\,(\mathrm{Ax})
}{\Phi,\psi\vdash\theta}\,(\Rightarrow\mathrm{E})
}{
\dfrac{\Phi\vdash\theta}{\varphi\Rightarrow\psi,\psi\Rightarrow\theta\vdash\varphi\Rightarrow\theta}\,(\Rightarrow\mathrm{I})
}\,(\mathrm{Cut})
$$

$(\Phi \triangleq \varphi\Rightarrow\psi, \psi\Rightarrow\theta, \varphi)$

Simply Typed Lambda Calculus (STLC)

(with finite products)

Simple types:

$$A, B, C, \ldots \; ::= \; G, G', \ldots$$

"ground" types

$\quad$ unit $\qquad$ unit type

$\quad A \times B \qquad$ product type

$\quad A \to B \qquad$ function type

Simply Typed Lambda Calculus (STLC)
(with finite products)

Terms:

$s, t, r, \ldots$ ::=

c^A

x

$()$

(s,t)

$\mathrm{fst}\ t$

$\mathrm{snd}\ t$

$\lambda x{:}A.t$

$s\,t$

Constants
each with
a given type

Variables
(countably
many)

Simple types:

$A, B, C, \ldots$ ::= $G, G', \ldots$

unit

$A \times B$

$A \to B$

λ-abstraction

application

Alpha Equivalence

STLC terms are abstract syntax trees **modulo renaming λ-bound variables.**

E.g. $\lambda f : A \to B.\ \lambda x : A.\ f x$

& $\lambda x : A \to B.\ \lambda y : A.\ x y$

are the same term.

Alpha Equivalence

STLC terms are abstract syntax trees modulo renaming λ-bound variables.

E.g. $\lambda f : A \to B . \lambda x : A . f x$

& $\lambda x : A \to B . \lambda y : A . x y$

are the same term.

Formally, we quotient syntax trees by the equivalence relation of α-equivalence $=_\alpha$

(or use a "nameless" (de Bruijn) representation).

Alpha Equivalence

$$\frac{}{c^A =_\alpha c^A} \qquad \frac{}{x =_\alpha x} \qquad \frac{}{() =_\alpha ()}$$

$$\frac{s =_\alpha s' \quad t =_\alpha t'}{(s,t) =_\alpha (s',t')} \qquad \frac{t =_\alpha t'}{\mathrm{fst}\, t =_\alpha \mathrm{fst}\, t'} \qquad \frac{t =_\alpha t'}{\mathrm{snd}\, t =_\alpha \mathrm{snd}\, t'}$$

$$\frac{s =_\alpha s' \quad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

result of replacing all occurrences of x with y in term t

$$\frac{(y\,x)\cdot t =_\alpha (y\,x')\cdot t' \quad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\, t =_\alpha \lambda x' : A.\, t'}$$

Simply Typed Lambda Calculus (STLC)

Typing relation $\Gamma \vdash t : A$

typing environment = finite list of (var, type)-pairs (comma separated "snoc" lists)

$\Gamma ::= \diamond \mid \Gamma, x : A$

(only the lists whose variables are distinct get used)

term

type

is inductively defined by the following rules...

$\boxed{\Gamma \text{ ok}}$ means: no variable occurs more than once in Γ

$\boxed{\mathrm{dom}\,\Gamma}$ = finite set of variables occurring in Γ

Typing rules for variables

$$\frac{\Gamma \text{ ok} \qquad x \notin \mathrm{dom}\,\Gamma}{\Gamma, x : A \vdash x : A}\ (\mathrm{var})$$

$$\frac{\Gamma \vdash x : A \qquad x' \notin \mathrm{dom}\,\Gamma}{\Gamma, x' : A' \vdash x : A}\ (\mathrm{var}')$$

$\boxed{\Gamma \text{ ok}}$ means: no variable occurs more than once in Γ

Typing rules for constants & unit value

$$\frac{\Gamma \text{ ok}}{\Gamma \vdash c^A : A}\ (\text{const})$$

$$\frac{\Gamma \text{ ok}}{\Gamma \vdash () : \text{unit}}\ (\text{unit})$$

Typing rules for pairing and projections

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash t' : A'}{\Gamma \vdash (t, t') : A \times A'}\ (\text{pair})$$

$$\frac{\Gamma \vdash t : A \times A'}{\Gamma \vdash \mathrm{fst}\ t : A}\ (\text{fst})$$

$$\frac{\Gamma \vdash t : A \times A'}{\Gamma \vdash \mathrm{snd}\ t : A'}\ (\text{snd})$$

Typing rules for function application & abstraction

$$\frac{\Gamma \vdash t : A \to A' \qquad \Gamma \vdash t' : A}{\Gamma \vdash t\,t' : A'}\ (\mathrm{app})$$

$$\frac{\Gamma, x : A \vdash t : A'}{\Gamma \vdash \lambda x : A.\, t : A \to A'}\ (\lambda)$$

Typing rules for function application & abstraction

$$\frac{\Gamma \vdash t : A \to A' \qquad \Gamma \vdash t' : A}{\Gamma \vdash t\,t' : A'}\ (\text{app})$$

$$\frac{\Gamma, x : A \vdash t : A'}{\Gamma \vdash \lambda x : A . t : A \to A'}\ (\lambda)$$

<u>N.B.</u> when using rule (λ) "bottom-up" to search for a proof of $\Gamma \vdash \lambda x : A . t : A \to A'$, since terms are syntax trees mod $=_\alpha$, can always assume $x \notin \mathrm{dom}\,\Gamma$

Example typing derivation

$$
\dfrac{
  \dfrac{\dfrac{}{\Gamma \vdash g : B \to C}\,(\mathrm{var})}{\Gamma, x : A \vdash g : B \to C}\,(\mathrm{var}')
  \qquad
  \dfrac{
    \dfrac{\vdots}{\Gamma, x : A \vdash f : A \to B}\,(\mathrm{var}')
    \qquad
    \dfrac{}{\Gamma, x : A \vdash x : A}\,(\mathrm{var})
  }{\Gamma, x : A \vdash f x : B}\,(\mathrm{app})
}{
  \dfrac{\Gamma, x : A \vdash g(f x) : C}{\Gamma \vdash \lambda x : A . g(f x) : A \to C}\,(\lambda)
}\,(\mathrm{app})
$$

(where $\Gamma \triangleq [\diamond, f : A \to B, g : B \to C]$)

<u>N.B.</u> typing rules are "syntax-directed" (by the structure of t & then Γ, for variables)

Semantics of STLC types in a ccc $\mathbb{C}$

Given a function M

ground types $G \mapsto$ objects $M(G) \in \mathbb{C}$

we extend it to a function

types $A \mapsto$ objects $M[\![A]\!] \in \mathbb{C}$

by recursion on the structure of A:

$$M[\![G]\!] = M(G)$$

$$M[\![\text{unit}]\!] = 1$$

terminal object

$$M[\![A \times B]\!] = M[\![A]\!] \times M[\![B]\!]$$

product

$$M[\![A \to B]\!] = M[\![B]\!]^{(M[\![A]\!])}$$

exponential

Semantics of STLC types in a ccc $\mathbb{C}$

$$\boxed{\begin{aligned} M[\![G]\!] &= M(G) \\ M[\![\mathrm{unit}]\!] &= 1 \\ M[\![A \times B]\!] &= M[\![A]\!] \times M[\![B]\!] \\ M[\![A \to B]\!] &= M[\![B]\!]^{(M[\![A]\!])} \end{aligned}}$$

extend this to typing environments:

$$M[\![\Diamond]\!] = 1$$

$$M[\![\Gamma, x : A]\!] = M[\![\Gamma]\!] \times M[\![A]\!]$$