Ccc

Definition A cartesian closed category (ccc) is a category $\mathbb{C}$ with

- a terminal object
- binary products
- an exponential for every pair of objects

Non-example of a ccc

Category of monoids **Mon** is not a ccc, because:

free monoid on $2 = \{0,1\}$

by univ. prop. of free monoid

$$\mathbb{N} \cong 2^* \times 2^* \cong \mathbf{Set}(2, 2^*) \cong \mathbf{Mon}(2^*, 2^*)$$

because $1 \times M \cong M$

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(Here I'm writing X^* instead of $\mathrm{List}(X)$ for the set of finite lists of elements of a set X.)

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so $\mathbf{Mon}(1 \times 2^*, 2^*) \not\cong \mathbf{Mon}(1, M)$ for any M, and hence the exponential of 2^* & 2^* can't exist in **Mon**.

Examples of ccc

A pre-ordered set $(X, \leq)$ regarded as a category is cartesian iff it has

- a greatest element $\top$: $(\forall p \in P)\ p \leq \top$
- binary meets $p \wedge q$: $(\forall r \in P)\ r \leq p \wedge q \Leftrightarrow r \leq p \ \&\ r \leq q$

It is a ccc iff it has

- Heyting implications $p \to q$:

$(\forall r \in P)\ r \leq p \to q \Leftrightarrow r \wedge p \leq q$

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E.g. any Boolean algebra ($p \to q = \neg p \vee q$),
Also $([0,1], \leq)$, for which $p \to q = \begin{cases} 1 & \text{if } p \leq q \\ q & \text{if } q < p \end{cases}$

Intuitionistic Propositional Logic

- "natural deduction" style
- only conjunction & implication fragment

Formulas:

$$
\begin{aligned}
\varphi, \psi, \theta, \ldots \; ::= \; & p, q, r, \ldots && \boxed{\text{propositional identifiers}} \\
& \top && \boxed{\text{truth}} \\
& \varphi \,\&\, \psi && \boxed{\text{conjunction}} \\
& \varphi \Rightarrow \psi && \boxed{\text{implication}}
\end{aligned}
$$

Intuitionistic Propositional Logic

Entailment relation $\Phi \vdash \varphi$

hypotheses, a finite list of formulas

conclusion, a formula

is inductively defined by the following rules:

(which use the notation Φ, φ for the finite list of formulas whose head is φ and whose tail is the list Φ)

Intuitionistic Propositional Logic

$$\frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\,(\mathrm{Cut})$$

$$\frac{}{\Phi, \varphi \vdash \varphi}\,(\mathrm{Ax})$$

$$\frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\,(\mathrm{Wk})$$

$$\frac{}{\Phi \vdash \top}\,(\top)$$

$$\frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \,\&\, \psi}\,(\wedge I)$$

$$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\,(\Rightarrow I)$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \varphi}\,(\wedge E_1)$$

$$\frac{\Phi \vdash \varphi \,\&\, \psi}{\Phi \vdash \psi}\,(\wedge E_2)$$

$$\frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\,(\Rightarrow E)$$

For example $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ holds:

$$\dfrac{\overbrace{\boxed{\varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi}}^{\Phi} \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)$$

For example $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ holds :

$$\frac{\Phi \vdash \theta}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)$$

$$\left(\Phi \triangleq \varphi \Rightarrow \psi,\ \psi \Rightarrow \theta,\ \varphi \right)$$

For example $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ holds :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \psi \Rightarrow \theta}\,(Ax)}{\Phi \vdash \psi \Rightarrow \theta}\,(Wk) \qquad \Phi \vdash \psi}{\Phi \vdash \theta}\,(\Rightarrow E)}{\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,(\Rightarrow I)$$

$$\left(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi \right)$$

For example $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ holds :

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \psi \Rightarrow \theta$}
\RightLabel{(Wk)}
\UnaryInfC{$\Phi \vdash \psi \Rightarrow \theta$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\cdots$}
\RightLabel{(Wk)}
\UnaryInfC{$\cdots$}
\RightLabel{(Wk)}
\UnaryInfC{$\Phi \vdash \varphi \Rightarrow \psi$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Phi \vdash \varphi$}
\RightLabel{$(\Rightarrow E)$}
\BinaryInfC{$\Phi \vdash \psi$}
\RightLabel{$(\Rightarrow E)$}
\BinaryInfC{$\Phi \vdash \theta$}
\RightLabel{$(\Rightarrow I)$}
\UnaryInfC{$\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$}
\end{prooftree}
$$

$$\left(\Phi \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi \right)$$

Semantics of IPL in a cartesian closed pre-order $(P, \leqslant)$

Given a meaning M for each propositional identifier p as an element $M_p \in P$, we get a semantics for formulas $M[\![\varphi]\!] \in P$:

$$M[\![p]\!] = M_p$$

$$M[\![\top]\!] = \top$$ greatest element

$$M[\![\varphi \,\&\, \psi]\!] = M[\![\varphi]\!] \wedge M[\![\psi]\!]$$ binary meet

$$M[\![\varphi \Rightarrow \psi]\!] = M[\![\varphi]\!] \to M[\![\psi]\!]$$ Heyting implication

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$$M[\![\varphi \Rightarrow \psi]\!] = M[\![\varphi]\!] \to M[\![\psi]\!] \quad \text{Heyting implication}$$

and a semantics for lists of formulas

$M[\![\Phi]\!] \in P$:

$$M[\![\emptyset]\!] = \top$$

$$M[\![\Phi, \varphi]\!] = M[\![\Phi]\!] \wedge M[\![\varphi]\!]$$

Semantics of IPL in a cartesian closed pre-order $(P, \leqslant)$

Soundness theorem

If $\Phi \vdash \varphi$ is provable from the rules of IPL, then $M[\![\Phi]\!] \leq M[\![\varphi]\!]$ holds in any cartesian closed pre-order.

Proof – exercise.

(show that $\{(\Phi, \varphi) \mid M[\![\Phi]\!] \leq M[\![\varphi]\!]\}$ is closed under the axioms & rules of IPL & hence contains $\{(\Phi, \varphi) \mid \Phi \vdash \varphi \text{ is provable}\}$)

Example

application of the Soundness Theorem :

Peirce's Law $\quad T \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$

is <u>not</u> provable in IPL

(whereas $((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ is a classical tautology)

because in the c.c. pre-order $([0,1], \leq)$

taking $Mp = \frac{1}{2}$, $Mq = 0$ we get

$$
\begin{aligned}
M[\![((p \Rightarrow q) \Rightarrow p) \Rightarrow p]\!] &= ((\tfrac{1}{2} \to 0) \to \tfrac{1}{2}) \to \tfrac{1}{2} \\
&= (0 \to \tfrac{1}{2}) \to \tfrac{1}{2} \\
&= 1 \to \tfrac{1}{2} \\
&= \tfrac{1}{2}
\end{aligned}
$$

Semantics of IPL in a cartesian closed poset $(P, \leqslant)$

Completeness Theorem

Given Φ, φ, if for all c.c. posets $(P, \leqslant)$ and all interpretations M of the propositional identifiers as elements of P, it is the case ~~that~~

$$M[\![\Phi]\!] \leqslant M[\![\varphi]\!]$$

in P, then $\Phi \vdash \varphi$ is provable in IPL.

Proof ...

Proof Define

$$P \triangleq \{\text{formulas of IPL}\}$$
$$\varphi \le \psi \triangleq \varphi \vdash \psi \text{ is provable in IPL}$$

Then $(P, \leq)$ is a c.c. pre-ordered set with an interpretation of IPL given by $M\,p = p$.

Can show that $M[\![\Phi]\!] \le M[\![\psi]\!]$ in this $(P, \le)$ iff $\Phi \vdash \psi$ is valid in IPL.

$\square$