L108 Assessment heads up

Assessed exercise sheet (ExSh#4)
(for 25% credit)

- issued Monday 7 Nov (in class)
- your answers are due by Monday 14 Nov, 16:00

(Take-home exam, 75% credit, in Jan.)

Exponentials

Given sets $X, Y \in \mathbf{Set}$, we have

$Y^X \in \mathbf{Set}$ set of all functions with domain X & codomain Y

$Y^X = \mathbf{Set}(X,Y) = \{ f \subseteq X \times Y \mid f \text{ is single-valued \& total} \}$

Aim to characterize Y^X category theoretically

Function application:

$$\mathrm{app} \in \mathbf{Set}(Y^X \times X, Y)$$

$$\mathrm{app}(f, x) = f x \qquad (f \in Y^X, x \in X)$$

so $\mathrm{app} \subseteq (Y^X \times X) \times Y$ is

$$\{((f, x), y) \mid (x, y) \in f\}$$

Function application:

$$\mathrm{app} \in \mathbf{Set}(Y^X \times X, Y)$$

$$\mathrm{app}(f, x) \triangleq f x \qquad (f \in Y^X, x \in X)$$

Function currying:

$$\frac{f \in \mathbf{Set}(Z \times X, Y)}{\mathrm{cur} f \in \mathbf{Set}(Z, Y^X)}$$

$$\mathrm{cur} f\, z\, x \triangleq f(z, x) \qquad (z \in Z, x \in X)$$

so $\mathrm{cur} f\, z = \{(x, y) \mid ((z, x), y) \in f\}$

Haskell Curry

Mathematician

Haskell Brooks Curry was an American mathematician and logician. Curry is best known for his work in combinatory logic; while the initial concept of combinatory logic was based on a single paper by ... Wikipedia

Born: September 12, 1900, Millis, Massachusetts, United States

Died: September 1, 1982, State College, Pennsylvania, United States

Parents: Samuel Silas Curry

Books: A Theory of Formal Deducibility, Foundations of Mathematical Logic

Education: University of Göttingen (1930), Harvard University

Function application:

$$\mathrm{app} \in \mathbf{Set}(Y^X \times X, Y)$$

$$\mathrm{app}(f, x) \triangleq f\,x \qquad (f \in Y^X, x \in X)$$

Function **currying**:

this notation just means "cur is a function from the set Set(Z×X, Y) to the set Set(Z, Y^X)"

$$\frac{f \in \mathbf{Set}(Z \times X, Y)}{\mathrm{cur}\, f \in \mathbf{Set}(Z, Y^X)}$$

$$\mathrm{cur}\, f\, z\, x \triangleq f(z, x) \qquad (z \in Z, x \in X)$$

so $\mathrm{cur}\, f\, z = \{(x, y) \mid ((z, x), y) \in f\}$

Given $f \in \mathbf{Set}(Z \times X, Y)$, get commutative diagram

$$
\begin{tikzcd}
Y^X \times X \arrow[r, "\mathrm{app}"] & Y \\
Z \times X \arrow[u, "\mathrm{cur}f \times \mathrm{id}_X"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{cur}f\, z, x) \arrow[r, mapsto] & \mathrm{cur}f\, z\, x = f(z,x) \\
(z,x) \arrow[u, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}
$$

$\boxed{\text{see Ex. Sheet 2, qu. 1(b)}}$

Given $f \in \mathbf{Set}(Z \times X, Y)$, get commutative diagram

$$
\begin{tikzcd}
Y^X \times X \arrow[r, "\mathrm{app}"] & Y \\
Z \times X \arrow[u, "\mathrm{cur} f \times \mathrm{id}_X"] \arrow[ur, "f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{cur} f\, z, x) \arrow[r, mapsto] & \mathrm{cur} f\, z\, x \\
 & f(z, x) \arrow[u, phantom, "\shortparallel"] \\
(z, x) \arrow[uu, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}
$$

Furthermore, if $g \in \mathbf{Set}(Z, Y^X)$ also satisfies

$$
\begin{tikzcd}
Y^X \times X \arrow[r, "\mathrm{app}"] & Y \\
Z \times X \arrow[u, "g \times \mathrm{id}_X"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

then $g = \mathrm{cur} f$,

because of function extensionality...

Function extensionality

Two functions $f, g \in Y^X$ are equal if (and only if)

$$(\forall x \in X)\ f x = g x$$

because this implies

$$\{(x, fx) \mid x \in X\} = \{(x, gx) \mid x \in X\}$$

ie $\{(x,y) \mid (x,y) \in f\} = \{(x,y) \mid (x,y) \in g\}$

i.e. $f = g$

Exponentials

in any category $\mathbb{C}$ that has binary products

so we assume ~~that~~ for every pair of objects X & Y in $\mathbb{C}$, we are given a product diagram for them

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

Exponentials

in any category $\mathbb{C}$ that has binary products

An exponential for $\mathbb{C}$-objects X & Y is specified by
object Y^X + morphism $app : Y^X \times X \to Y$
with the universal property:

for all $f \in \mathbb{C}(Z \times X, Y)$ there is a unique morphism $g \in \mathbb{C}(Z, Y^X)$ such that

$$
\begin{tikzcd}
Y^X \times X \arrow[r, "app"] & Y \\
Z \times X \arrow[u, "g \times id_X"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

commutes

Exponentials

An **exponential** for $\mathbb{C}$-objects X & Y is specified by

object Y^X + morphism $app : Y^X \times X \to Y$

with the universal property:

for all $f \in \mathbb{C}(Z \times X, Y)$ there is a unique morphism $g \in \mathbb{C}(Z, Y^X)$ such that

$$\begin{tikzcd} Y^X \times X \arrow[r, "app"] & Y \\ Z \times X \arrow[u, "g \times id_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes

Notation: we'll write $cur\, f$ for this unique g

Exponentials

The universal property of $\mathrm{app} : Y^X \times X \to Y$ says that there is a bijection:

$$\boxed{\begin{array}{ccc} \mathbb{C}(Z, Y^X) & \cong & \mathbb{C}(Z \times X, Y) \\ g & \longmapsto & \mathrm{app} \circ (g \times id_X) \\ \mathrm{cur}\, f & \longleftarrow\!\!\! \shortmid & f \\ \\ \multicolumn{3}{c}{\mathrm{app} \circ (\mathrm{cur}\, f \times id_X) = f} \\ \multicolumn{3}{c}{\mathrm{cur}(\mathrm{app} \circ (g \times id_X)) = g} \end{array}}$$

Exponentials

An **exponential** for $\mathbb{C}$-objects X & Y is specified by

object Y^X + morphism $app : Y^X \times X \to Y$

such that

(Y^X, app) is terminal in the category with

- objects (Z, f) where $f \in \mathbb{C}(Z \times X, Y)$
- morphisms $g : (Z, f) \to (Z', f')$ are $g \in \mathbb{C}(Z, Z')$ such that $f' \circ (g \times id_X) = f$
- composition & identities as in $\mathbb{C}$

Exponentials

An **exponential** for $\mathbb{C}$-objects X & Y is specified by object Y^X + morphism $\mathrm{app} : Y^X \times X \to Y$ such that

$\boxed{\begin{array}{l} (Y^X, \mathrm{app}) \text{ is terminal in the category with} \\ \text{– objects } (Z, f) \text{ where } f \in \mathbb{C}(Z \times X, Y) \\ \text{– morphisms } g : (Z, f) \to (Z', f') \text{ are } g \in \mathbb{C}(Z, Z') \\ \quad \text{such that } f' \circ (g \times id_X) = f \\ \text{– composition \& identities as in } \mathbb{C} \end{array}}$

so if they exist, exponentials are unique up to (unique) isomorphism

Ccc's

<u>Definition</u> A cartesian closed category (ccc) is a category $\mathbb{C}$ with

- a terminal object
- binary products
- an exponential for every pair of objects

Examples of ccc's

- **Set** is a ccc – as we've seen.
- **Pre** is a ccc: the exponential of $(P, \leq)$ and $(Q, \leq)$ is $(P \to Q, \leq)$ where

$$P \to Q = \{ f \in Q^P \mid (\forall p, p' \in P)\ p \leq p' \Rightarrow fp \leq fp' \}$$

$\boxed{\text{this is just } \mathbf{Pre}((P, \leq), (Q, \leq))}$