L108 Assessment heads up

Assessed exercise sheet (ExSh#4) (for 25% credit)

- issued Monday 7 Nov (in class)
- your answers are due by Monday 14 Nov, 16:00

(Take-home exam, 75% credit, in Jan.)
Exponentials

Given sets \( X, Y \in \text{Set} \), we have

\[ Y^X \in \text{Set} \quad \text{set of all functions with domain } X \text{ and codomain } Y \]

\[ Y^X = \text{Set}(X, Y) = \{ f \subseteq X \times Y | f \text{ is single-valued } \}
\]

Aim to characterize \( Y^X \)

Category theoretically
Function application:

\[ \text{app} \in \text{Set}(Y^X \times X, Y) \]

\[ \text{app}(f, x) = f x \quad (f \in Y^X, x \in X) \]

So \( \text{app} \subseteq (Y^X \times X) \times Y \) is

\[ \{((f, x), y) \mid (x, y) \in f \} \]
Function application:

\[ \text{app} \in \text{Set}(Y^X \times X, Y) \]
\[ \text{app}(f, x) \triangleq fx \quad (f \in Y^X, x \in X) \]

Function currying:

\[ f \in \text{Set}(Z \times X, Y) \]
\[ \text{currf} \in \text{Set}(Z, Y^X) \]
\[ \text{currf} z x \triangleq f(z, x) \quad (z \in Z, x \in X) \]

so \( \text{currf} z = \{ (x, y) \mid ((z, x), y) \in f \} \)
Haskell Curry was an American mathematician and logician. Curry is best known for his work in combinatory logic; while the initial concept of combinatory logic was based on a single paper by ...

**Born:** September 12, 1900, Millis, Massachusetts, United States

**Died:** September 1, 1982, State College, Pennsylvania, United States

**Parents:** Samuel Silas Curry

**Books:** A Theory of Formal Deducibility, Foundations of Mathematical Logic

**Education:** University of Göttingen (1930), Harvard University
**Function application:**

\[ \text{app} \in \text{Set}(Y^X \times X, Y) \]

\[ \text{app}(f, x) \triangleq f_x \quad (f \in Y^X, x \in X) \]

**Function currying:**

\[ f \in \text{Set}(Z \times X, Y) \]

\[ \text{curf} \in \text{Set}(Z, Y^X) \]

\[ \text{curf} z x \triangleq f(z, x) \quad (z \in Z, x \in X) \]

\[ \text{so} \; \text{curf} z = \{ (x, y) \mid ((z, x), y) \in f \} \]
Given \( f \in \text{Set}(Z \times X, Y) \), get commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{app}} & Y \\
\uparrow{\text{curf}} \times \text{id}_X & & \\
Z \times X & \xrightarrow{f} & Z \times X
\end{array}
\]

see Ex. Sheet 2, q. 1(b)
Given \( f \in \text{Set}(\mathbb{Z} \times \mathbb{X}, \mathbb{Y}) \), get commutative diagram

\[
\begin{array}{c}
\mathbb{Y} \times \mathbb{X} \times \mathbb{X} \xrightarrow{\text{app}} \mathbb{Y} \\
\text{curf} \times \text{id}_x \uparrow \\
\mathbb{Z} \times \mathbb{X} \times \mathbb{X} \xrightarrow{f}
\end{array}
\]

Furthermore, if \( g \in \text{Set}(\mathbb{Z}, \mathbb{Y}^\mathbb{X}) \) also satisfies

\[
\begin{array}{c}
\mathbb{Y} \times \mathbb{X} \times \mathbb{X} \xrightarrow{\text{app}} \mathbb{Y} \\
g \times \text{id}_x \uparrow \\
\mathbb{Z} \times \mathbb{X} \times \mathbb{X} \xrightarrow{f}
\end{array}
\]

then \( g = \text{curf} \), because of function extensionality...
Function extensionality

Two functions $f, g \in Y^X$ are equal if (and only if)

$$(\forall x \in X) \ f x = g x$$

because this implies

$$\{ (x, f x) \mid x \in X \} = \{ (x, g x) \mid x \in X \}$$

i.e.

$$\{ (x, y) \mid (x, y) \in f \} = \{ (x, y) \mid (x, y) \in g \}$$

i.e.

$f = g$
Exponentials in any category $\mathcal{C}$ that has binary products so we assume that for every pair of objects $X$ and $Y$ in $\mathcal{C}$, we are given a product diagram for them

$$X \leftarrow^{\pi_1} X \times Y \rightarrow^{\pi_2} Y$$
Exponentials

in any category \( C \) that has binary products

An exponential for \( C \)-objects \( X \) & \( Y \)
is specified by

object \( Y^X \) + morphism \( \text{app} : Y^X \times X \to Y \)

with the universal property:

for all \( f \in C(Z \times X, Y) \) there is

a unique morphism \( g \in C(Z, Y^X) \)
such that \( Y^X \times X \xrightarrow{\text{app}} Y \)

commutes

\[ g \times \text{id}_X \]

\[ Z \times X \xrightarrow{f} \]
Exponentials

An exponential for $C$-objects $X$ & $Y$ is specified by an object $Y^X$ + morphism $\text{app}: Y^X \times X \to Y$ with the universal property:

For all $f \in C(Z \times X, Y)$ there is a unique morphism $g \in C(Z, Y^X)$ such that $Y^X \times X \xrightarrow{\text{app}} Y$ commutes.

Notation: we'll write $\text{curf}$ for this unique $g$. 
Exponentials

The universal property of $\text{app} : Y^X \times X \to Y$ says that there is a bijection:

\[ \mathcal{C}(Z, Y^X) \cong \mathcal{C}(Z \times X, Y) \]

\[ \begin{array}{c}
g \ \leftarrow \ \text{curf} \\
\text{app} \circ (g \times \text{id}_X) \leftrightarrow f \end{array} \]

\[ \text{app} \circ (\text{curf} \times \text{id}_X) = f \]

\[ \text{curf} \circ (\text{app} \circ (g \times \text{id}_X)) = g \]
Exponentials

An exponential for $C$-objects $X$ & $Y$ is specified by

object $Y^X$ + morphism $\text{app} : Y^X \times X \rightarrow Y$ such that

$(Y^X, \text{app})$ is terminal in the category with

- objects $(Z,f)$ where $f \in C(Z \times X, Y)$
- morphisms $g : (Z,f) \rightarrow (Z',f')$ are $g \in C(Z,Z')$ such that $f' \circ (g \times \text{id}_X) = f$
- composition & identities as in $C$
Exponentials

An exponential for $C$-objects $X$ & $Y$ is specified by

object $Y^X$ + morphism $\text{app} : Y^X \times X \to Y$

such that

$$(Y^X, \text{app})$$ is terminal in the category with
- objects $(Z, f)$ where $f \in C(Z \times X, Y)$
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- composition & identities as in $C$

so if they exist, exponentials are unique up to (unique) isomorphism.
Definition A cartesian closed category (ccc) is a category $\mathbf{C}$ with

- a terminal object
- binary products
- an exponential for every pair of objects
Examples of ccc's

- Set is a ccc - as we've seen.
- Pre is a ccc: the exponential of \((P, \leq)\) and \((Q, \leq)\) is \((P \to Q, \leq)\) where

\[
P \to Q = \{ f \in Q^P \mid (\forall p, p' \in P) \ p \leq p' \Rightarrow fp \leq fp' \}\]

this is just \(\text{Pre}((P, \leq), (Q, \leq))\)