Assessment heads-up

Take-home test (75%): collect from Grad. Office 4pm Thurs 19 Jan 2017.
Solutions due by 4pm Mon 23 Jan 2017.

Graded exercise Sheet (25%)
If you scored < 60% on Ex.Sh. 4 then you can have Ex.Sh. 6 graded.
HAND IN SOLUTIONS (to Grad. Office) by
4pm Mon 5 Dec 2016

Yoneda Lemma

For each small category $\mathbb{C}$, each $X \in \mathbb{C}$ and each $F \in \mathbf{Set}^{\mathbb{C}^{op}}$, there is a bijection of sets

$$\eta_{X,F} : \underbrace{\mathbf{Set}^{\mathbb{C}^{op}}(y(X), F)}_{\text{the set of natural transformations from the functor } y(X) : \mathbb{C}^{op} \to \mathbf{Set} \text{ to the functor } F : \mathbb{C}^{op} \to \mathbf{Set}} \cong \underbrace{F(X)}_{\text{the value of } F : \mathbb{C}^{op} \to \mathbf{Set} \text{ at } X}$$

which is natural in both X and F.

$$\boxed{\eta_{X,F} : \mathrm{Set}^{\mathbb{C}^{op}}(y(X), F) \longrightarrow F(X)}$$

Given $\theta : y(X) \to F$ in $\mathrm{Set}^{\mathbb{C}^{op}}$

we get $\underset{\mathbb{C}(X,X)}{\overset{\parallel}{y(X)X}} \xrightarrow{\theta_X} F(X)$

Define $\boxed{\eta_{X,F}(\theta) \triangleq \theta_X(id_X)}$

$$\boxed{\eta^{-1}_{X,F} : F(X) \longrightarrow \mathbf{Set}^{\mathbb{C}^{op}}(y(X), F)}$$

Given $x \in F(X)$,

for each $Y \in \mathbb{C}$ & $f \in \mathbb{C}(Y, X) = y(X)(Y)$

we get $F(f) : F(X) \to F(Y)$ in $\mathbf{Set}$

and hence $F(f)(x) \in F(Y)$

$$\boxed{\eta^{-1}_{X,F} : F(X) \to \mathbf{Set}^{\mathbb{C}^{op}}(y(X), F)}$$

Given $x \in F(X)$,

for each $Y \in \mathbb{C}$ & $f \in \mathbb{C}(Y, X) = y(X)(Y)$

we get $F(f) : F(X) \to F(Y)$ in $\mathbf{Set}$

and hence $F(f)(x) \in F(Y)$

Define $(\eta^{-1}_{X,F}\, x)_Y : y(X)Y \longrightarrow F(Y)$

to be the function $f \longmapsto F(f)(x)$

and check this gives a natural transformation

$$\eta^{-1}_{X,F}\, x : y(X) \to F$$

Proof of $\boxed{\eta_{X,F} \circ \eta_{X,F}^{-1} = id_{F(X)}}$

For any $x \in F(X)$

$$\eta_{X,F}\left(\eta_{X,F}^{-1} x\right) \overset{\Delta}{=} \left(\eta_{X,F}^{-1} x\right)_X (id_X)$$

definition of $\eta_{X,F}$

.

Proof of $\boxed{\eta_{X,F} \circ \eta^{-1}_{X,F} = id_{F(X)}}$

For any $x \in F(X)$

$$\eta_{X,F}(\eta^{-1}_{X,F}\, x) \triangleq (\eta^{-1}_{X,F}\, x)_X (id_X)$$

$$\triangleq F(id_X)(x)$$

definition of $\eta^{-1}_{X,F}$

Proof of $\boxed{\eta_{X,F} \circ \eta^{-1}_{X,F} = id_{F(X)}}$

For any $x \in F(X)$

$$\eta_{X,F}(\eta^{-1}_{X,F}\,x) \overset{\Delta}{=} (\eta^{-1}_{X,F}\,x)_X\,(id_X)$$

$$\overset{\Delta}{=} F(id_X)(x)$$

F is a functor $\longrightarrow$ $$= id_{F(X)}(x)$$

$$= x$$

$\square$

16:7

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathrm{id}_{\mathbf{Set}^{\mathcal{C}^{op}}(yX,F)}}$

For any $\left\{ \begin{array}{l} y(X) \xrightarrow{\theta} F \text{ in } \mathbf{Set}^{\mathcal{C}^{op}} \\ Y \xrightarrow{f} X \text{ in } \mathcal{C} \end{array} \right.$ we have

$$\eta^{-1}_{X,F}\left(\eta_{X,F}\,\theta\right) \triangleq \eta^{-1}_{X,F}\left(\theta_X(\mathrm{id}_X)\right)$$

definition of $\eta_{X,F}$

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathrm{id}_{\mathbf{Set}^{\mathcal{C}^{op}}(yX, F)}}$

For any $\begin{cases} y(X) \xrightarrow{\theta} F \text{ in } \mathbf{Set}^{\mathcal{C}^{op}} \\ Y \xrightarrow{f} X \text{ in } \mathcal{C} \end{cases}$ we have

$$\left(\eta^{-1}_{X,F}\left(\eta_{X,F}\,\theta\right)\right)_Y f \triangleq \left(\eta^{-1}_{X,F}\left(\theta_X(\mathrm{id}_X)\right)\right)_Y f$$

definition of $\eta^{-1}_{X,F}$

$$\triangleq F(f)\left(\theta_X(\mathrm{id}_X)\right)$$

Proof of $\boxed{\eta^{-1}_{X,F}\circ\eta_{X,F} = \mathrm{id}_{\mathbf{Set}^{\mathbb{C}^{op}}(yX,F)}}$

For any $\left\{\begin{array}{l} y(X)\xrightarrow{\theta} F \text{ in } \mathbf{Set}^{\mathbb{C}^{op}} \\ Y \xrightarrow{f} X \text{ in } \mathbb{C}\end{array}\right.$ we have

$$\begin{aligned}
\left(\eta^{-1}_{X,F}\left(\eta_{X,F}\,\theta\right)\right)_Y f &\triangleq \left(\eta^{-1}_{X,F}\left(\theta_X(\mathrm{id}_X)\right)\right)_Y f \\
&\triangleq F(f)\left(\theta_X(\mathrm{id}_X)\right) \\
&= \theta_Y\left(f^*(\mathrm{id}_X)\right)
\end{aligned}$$

$$\boxed{\begin{tikzcd}
y(X)Y \arrow[r, "\theta_Y"] & F(Y) \\
y(X)X \arrow[u, "f^*"] \arrow[r, "\theta_X"'] \arrow[ur, phantom, "\text{naturality of } \theta"] & F(X) \arrow[u, "F(f)"']
\end{tikzcd}}$$

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathrm{id}_{\mathbf{Set}^{\mathcal{C}^{op}}(y X, F)}}$

For any $\begin{cases} y(X) \xrightarrow{\theta} F \text{ in } \mathbf{Set}^{\mathcal{C}^{op}} \\ Y \xrightarrow{f} X \text{ in } \mathcal{C} \end{cases}$ we have

$$\begin{aligned}
\left(\eta^{-1}_{X,F}\left(\eta_{X,F}\,\theta\right)\right)_Y f &\triangleq \left(\eta^{-1}_{X,F}\left(\theta_X(\mathrm{id}_X)\right)\right)_Y f \\
&\triangleq F(f)\left(\theta_X(\mathrm{id}_X)\right) \\
&= \theta_Y\left(f^*(\mathrm{id}_X)\right) \\
&\triangleq \theta_Y\left(\mathrm{id}_X \circ f\right) \\
&= \theta_Y(f)
\end{aligned}$$

definition of f^*

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathrm{id}_{\mathbf{Set}^{\mathcal{C}^{op}}(y X, F)}}$

so for all θ, Y, f

$$\left(\eta^{-1}_{X,F}\left(\eta_{X,F}\,\theta\right)\right)_Y f = \theta_Y(f)$$

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathrm{id}_{\mathbf{Set}^{\mathbb{C}^{op}}(yX,F)}}$

so for all θ, Y

$$\left(\eta^{-1}_{X,F}\left(\eta_{X,F}\,\theta\right)\right)_Y = \theta_Y$$

16.13

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathrm{id}_{\mathbf{Set}^{\mathcal{C}^{op}}(y^X, F)}}$

so for all θ,

$$\eta^{-1}_{X,F}\left(\eta_{X,F}\,\theta\right) = \theta$$

$\square$

16:14

Yoneda Lemma

For each small category $\mathbb{C}$, each $X \in \mathbb{C}$ and each $F \in \mathbf{Set}^{\mathbb{C}^{op}}$, there is a bijection of sets

$$\eta_{X,F} : \underbrace{\mathbf{Set}^{\mathbb{C}^{op}}(y(X), F)}_{\substack{\text{the set of natural transformations} \\ \text{from the functor } y(X) : \mathbb{C}^{op} \to \mathbf{Set} \\ \text{to the functor } F : \mathbb{C}^{op} \to \mathbf{Set}}} \cong \underbrace{F(X)}_{\substack{\text{the value of} \\ F : \mathbb{C}^{op} \to \mathbf{Set} \\ \text{at } X}}$$

which is natural in both X and F.

Proof that $\eta_{X,F}$ is natural in X :

given $\begin{tikzcd} X \\ X' \arrow[u, "f"] \end{tikzcd}$ does

$$\begin{tikzcd} \mathrm{Set}^{\mathcal{C}^{op}}(y(X),F) \arrow[r, "\eta_{X,F}"] \arrow[d, "y(f)_*"'] & F(X) \arrow[d, "F(f)"] \\ \mathrm{Set}^{\mathcal{C}^{op}}(y(X'),F) \arrow[r, "\eta_{X',F}"'] & F(X') \end{tikzcd}$$

commute ?

Proof that $\eta_{X,F}$ is natural in X:

given $\begin{tikzcd} X \\ X' \arrow[u, "f"] \end{tikzcd}$ does $\begin{tikzcd} \mathrm{Set}^{\mathcal{C}^{op}}(y(X),F) \arrow[r, "\eta_{X,F}"] \arrow[d, "y(f)_*"'] & F(X) \arrow[d, "F(f)"] \\ \mathrm{Set}^{\mathcal{C}^{op}}(y(X'),F) \arrow[r, "\eta_{X',F}"'] & F(X') \end{tikzcd}$ commute?

For all $\theta : y(X) \to F$ have

$$F(f)(\eta_{X,F}\,\theta) \triangleq F(f)(\theta_X(id_X))$$

$$= \theta_{X'}(f^*(id_X))$$

$$= \theta_{X'}(f)$$

by naturality of θ:

$$\begin{tikzcd} y(X)X \arrow[r, "\theta_X"] \arrow[d, "f^*"'] & F(X) \arrow[d, "F(f)"] \\ y(X)X' \arrow[r, "\theta_{X'}"'] & F(X') \end{tikzcd}$$

$$f^*(id) \triangleq id \circ f = f$$

16.17

Proof that $\eta_{X,F}$ is natural in X:

given
$$\begin{tikzcd} X \\ X' \arrow[u, "f"] \end{tikzcd}$$
does
$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}^{op}}(y(X),F) \arrow[r, "\eta_{X,F}"] \arrow[d, "y(f)_*"'] & F(X) \arrow[d, "F(f)"] \\
\mathbf{Set}^{\mathcal{C}^{op}}(y(X'),F) \arrow[r, "\eta_{X',F}"'] & F(X')
\end{tikzcd}$$
commute?

For all $\theta : y(X) \to F$ have

$$\begin{aligned}
F(f)(\eta_{X,F}\,\theta) &\doteq F(f)(\theta_X(id_X)) \\
&= \theta_{X'}(f^*(id_X)) \\
&= \theta_{X'}(f)
\end{aligned}$$

$$\begin{aligned}
&\eta_{X',F}(y(f)_*(\theta)) \\
&= \eta_{X',F}(\theta \circ y(f)) \\
&= (\theta \circ y(f))_{X'}(id_{X'}) \\
&= \theta_{X'}(y(f)_{X'}(id_{X'})) \\
&= \theta_{X'}(f_*(id_{X'}))
\end{aligned}$$

✓

16.18

Proof that $\eta_{X,F}$ is natural in F :

given $\begin{array}{c} F \\ \varphi\big\downarrow \\ G \end{array}$ does

$$\begin{array}{ccc} \mathrm{Set}^{\mathbb{C}^{op}}(y(X), F) & \xrightarrow{\eta_{X,F}} & F(X) \\ \varphi_* \big\downarrow & & \big\downarrow \varphi_X \\ \mathrm{Set}^{\mathbb{C}^{op}}(y(X), G) & \xrightarrow[\eta_{X,G}]{} & G(X) \end{array}$$

commute?

16.29

Proof that $\eta_{X,F}$ is natural in F:

given $\begin{array}{c} F \\ \varphi \downarrow \\ G \end{array}$ does

$$\begin{tikzcd}
\mathbf{Set}^{\mathbb{C}^{op}}(y(X),F) \arrow[r, "\eta_{X,F}"] \arrow[d, "\varphi_{*}"'] & F(X) \arrow[d, "\varphi_X"] \\
\mathbf{Set}^{\mathbb{C}^{op}}(y(X),G) \arrow[r, "\eta_{X,G}"'] & G(X)
\end{tikzcd}$$

commute?

For all $\theta : y(X) \to F$ have

$$\begin{aligned}
\varphi_X(\eta_{X,F}(\theta)) &\triangleq \varphi_X(\theta_X(id_X)) \\
&\triangleq (\varphi \circ \theta)_X(id_X) \\
&\triangleq \eta_{X,G}(\varphi \circ \theta) \\
&\triangleq \eta_{X,G}(\varphi_{*}(\theta)) \quad ✓
\end{aligned}$$

16·20

Proof that $\eta_{X,F}$ is natural in F :

given $\begin{array}{c} F \\ \varphi \downarrow \\ G \end{array}$ does

$$\begin{tikzcd} \mathbf{Set}^{\mathbb{C}^{op}}(y(X),F) \arrow[r, "\eta_{X,F}"] \arrow[d, "\varphi_*"'] & F(X) \arrow[d, "\varphi_X"] \\ \mathbf{Set}^{\mathbb{C}^{op}}(y(X),G) \arrow[r, "\eta_{X,G}"'] & G(X) \end{tikzcd}$$

commute?

For all $\theta : y(X) \to F$ have

$$\begin{aligned} \varphi_X(\eta_{X,F}(\theta)) &\triangleq \varphi_X(\theta_X(id_X)) \\ &\triangleq (\varphi \circ \theta)_X(id_X) \\ &\triangleq \eta_{X,G}(\varphi \circ \theta) \\ &\triangleq \eta_{X,G}(\varphi_*(\theta)) \end{aligned}$$

$\boxed{\text{this completes the proof of the Yoneda Lemma}}$

Corollary of the Yoneda Lemma:

$$y : \mathbb{C} \longrightarrow \mathbf{Set}^{\mathbb{C}^{op}}$$

is a full & faithful functor

In general, a functor $F : \mathbb{C} \to \mathbb{D}$ is **faithful** if for all $X, X' \in \mathbb{C}$

$$\begin{aligned} \mathbb{C}(X, X') &\longrightarrow \mathbb{D}(FX, FX') \\ f &\longmapsto F(f) \end{aligned}$$

is injective (one-one)

and **full** if those functions are surjective. (onto)

Corollary of the Yoneda Lemma:

$$y : \mathbb{C} \longrightarrow \mathrm{Set}^{\mathbb{C}^{op}}$$

is a full & faithful functor

Proof For all $X, X' \in \mathbb{C}$, from the proof of the Yoneda Lemma we have that

$$\begin{array}{ccc} \mathbb{C}(X, X') & \longrightarrow & \mathrm{Set}^{\mathbb{C}^{op}}(y(X), y(X')) \\ f & \longmapsto & y(f) \end{array}$$

is equal to

$$\mathbb{C}(X, X') \triangleq y(X')(X) \xrightarrow{(\eta_{X,F})^{-1}} \mathrm{Set}^{\mathbb{C}^{op}}(y(X), F)$$

when $F = y(X')$, and hence is a bijection, i.e. $\begin{cases} \text{injective \&} \\ \text{surjective} \end{cases}$

Theorem

For each small category $\mathbb{C}$, the category $\mathbf{Set}^{\mathbb{C}^{op}}$ is cartesian closed

Theorem
For each small category $\mathbb{C}$, the category $\mathbf{Set}^{\mathbb{C}^{op}}$ is cartesian closed

Proof (sketch)
Terminal object in $\mathbf{Set}^{\mathbb{C}^{op}}$ is functor $T : \mathbb{C}^{op} \to \mathbf{Set}$ given by $T(x) = \{*\}$ & $T(f) = id_{\{*\}}$

terminal in Set

Theorem

For each small category $\mathbb{C}$, the category $\mathbf{Set}^{\mathbb{C}^{op}}$ is cartesian closed

Proof (sketch)

Product of $F, G \in \mathbf{Set}^{\mathbb{C}^{op}}$ is

$$(F \times G)(X) = F(X) \times G(X)$$

$$(F \times G)(f) = F(f) \times G(f)$$

product in Set

with projection morphisms $F \xleftarrow{\pi_1} F \times G \xrightarrow{\pi_2} G$ given by

$(\pi_1)_X =$ fst proj^n, $(\pi_2)_X =$ snd proj^n

Theorem

For each small category $\mathbb{C}$, the category $\mathbf{Set}^{\mathbb{C}^{op}}$ is cartesian closed

Proof (sketch)

Exponential of $F, G \in \mathbf{Set}^{\mathbb{C}^{op}}$ is suggested by the Yoneda Lemma

Yoneda

$$G^F(X) \cong \mathbf{Set}^{\mathbb{C}^{op}}(y(X), G^F)$$

$$\cong \mathbf{Set}^{\mathbb{C}^{op}}(y(X) \times F, G)$$

universal property of G^F

Theorem

For each small category $\mathbb{C}$, the category $\mathbf{Set}^{\mathbb{C}^{op}}$ is cartesian closed

Proof (sketch)

Exponential of $F, G \in \mathbf{Set}^{\mathbb{C}^{op}}$ is suggested by the Yoneda Lemma

$$G^F(X) \cong \mathbf{Set}^{\mathbb{C}^{op}}(y(X), G^F)$$

$$\cong \boxed{\mathbf{Set}^{\mathbb{C}^{op}}(y(X) \times F, G)}$$

We take this as the definition of $G^F(X)$

$$\boxed{G^F(X) \triangleq \mathbf{Set}^{\mathbb{C}^{op}}(y(X) \times F, G)}$$

Given $X' \xrightarrow{f} X$ in $\mathbb{C}$

get $y(X') \xrightarrow{y(f)} y(X)$ in $\mathbf{Set}^{\mathbb{C}^{op}}$

and hence

$$G^F(X') \triangleq \mathbf{Set}^{\mathbb{C}^{op}}(yX' \times F, G) \to \mathbf{Set}^{\mathbb{C}^{op}}(yX \times F, G) \triangleq G^F(X)$$

$$\theta \longmapsto \theta \circ (y(f) \times id_F)$$

$$\boxed{G^F(x) \triangleq \mathbf{Set}^{\mathbb{C}^{op}}(y(x) \times F, G)}$$

Given $x' \xrightarrow{f} x$ in $\mathbb{C}$

get $y(x') \xrightarrow{y(f)} y(x)$ in $\mathbf{Set}^{\mathbb{C}^{op}}$

and hence

$$G^F(x') \triangleq \mathbf{Set}^{\mathbb{C}^{op}}(yX' \times F, G) \to \mathbf{Set}^{\mathbb{C}^{op}}(yX \times F, G) \triangleq G^F(x)$$

$$\theta \longmapsto \theta \circ (y(f) \times id_F)$$

We define:

$$\boxed{G^F(f) \triangleq (y(f) \times id_F)^*}$$

Have to check that these definitions make G^F into a functor $\mathbb{C}^{op} \to \mathbf{Set}$.

Given $F, G \in \mathbf{Set}^{\mathbb{C}^{op}}$, the application morphism

$$app : G^F \times F \longrightarrow G$$

is given by (for each object $X \in \mathbb{C}$)

$$(G^F \times F)(X) = G^F(X) \times F(X)$$

$$
\begin{tikzcd}
= \mathbf{Set}^{\mathbb{C}^{op}}(yX \times F, G) \times F(X) \arrow[d, "app_X"] & (\theta, x) \arrow[d, mapsto] \\
G(X) & \theta_X(id_X, x)
\end{tikzcd}
$$

$$\boxed{app_X(\theta, x) \triangleq \theta_X(id_X, x)}$$

Have to check that this is natural in $X \in \mathbb{C}$

Currying: $\dfrac{\theta : H \times F \to G}{\mathrm{cur}\,\theta : H \to G^F}$

$(\mathrm{cur}\,\theta)_X : H(X) \to G^F(X) = \mathbf{Set}^{\mathbb{C}^{op}}(yX \times F, G)$

maps each $z \in H(X)$ to the morphism

$(\mathrm{cur}\,\theta)_X z : yX \times F \to G$ in $\mathbf{Set}^{\mathbb{C}^{op}}$

whose component at $Y \in \mathbb{C}$ is the function

$(yX \times F)(Y) = \mathbb{C}(Y, X) \times F(Y) \longrightarrow G(Y)$

given by

$$\boxed{((\mathrm{cur}\,\theta)_X z)_Y (f, y) \triangleq \theta_Y (H(f) z, y)}$$

Currying: $\dfrac{\theta : H \times F \to G}{\mathrm{cur}\,\theta : H \to G^F}$

$$\boxed{((\mathrm{cur}\,\theta)_X z)_Y (f, y) \triangleq \theta_Y (H(f) z, y)}$$

Have to check ~~that~~ this is natural in Y, ~~then~~ ~~that~~ $(\mathrm{cur}\,\theta)_X$ is natural in X, then ~~that~~ $\mathrm{cur}\,\theta$ is the unique morphism $H \xrightarrow{\varphi} G^F$ satisfying $\mathrm{app} \circ (\varphi \times \mathrm{id}_F) = \theta$

(i.e. there's lots to check, but it's all routine!)

Current themes in CT

- semantics of effects in prog. langs
- higher dimensional category theory
  - Homotopy Type Theory
  - structural aspects of quantum comp / inf. theory

Advert: Jamie Vicary seminar tomorrow @ 14:15, room MR5 at C.M.S.