Equivalence

Two categories $\mathbb{C}$ & $\mathbb{D}$ are isomorphic if they are isomorphic objects in the category of categories (of some size), that is, there are

functors

$$\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$$

satisfying

$$\mathrm{Id}_{\mathbb{C}} = G \circ F$$
$$F \circ G = \mathrm{Id}_{\mathbb{D}}$$

(in which case, as usual, we write $\mathbb{C} \cong \mathbb{D}$)

Equivalence

Two categories $\mathbb{C}$ & $\mathbb{D}$ are equivalent if there are

functors & natural isomorphisms

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

$$\eta : \mathrm{Id}_{\mathbb{C}} \cong G \circ F$$

$$\varepsilon : F \circ G \cong \mathrm{Id}_{\mathbb{D}}$$

in which case one writes $\boxed{\mathbb{C} \simeq \mathbb{D}}$

Some deep results in mathematics take the form of equivalences:

E.g.

Stone duality: $\begin{pmatrix}\text{Category} \\ \text{of Boolean} \\ \text{algebras}\end{pmatrix}^{op} \simeq$ Category of compact, tot. disconn. Hausdorff spaces

Gelfand duality: $\begin{pmatrix}\text{abelian} \\ C^*\text{ algebras}\end{pmatrix}^{op} \simeq$ Compact Hausdorff Spaces

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

$\mathbf{Set}/I$ is a slice category [Ex. Sh. 4, qu. 6]

- objects are (X, f) where $f \in \mathbf{Set}(X, I)$
- morphisms $g : (X, f) \to (X', f')$ are $g \in \mathbf{Set}(X, X')$ satisfying $f' \circ g = f$ in $\mathbf{Set}$
- composition & identities – as for $\mathbf{Set}$

For each $I \in \operatorname{obj} \mathbf{Set}$, let $\boxed{\mathbf{Set}^I}$ be the category with

- $\operatorname{obj}(\mathbf{Set}^I) \triangleq (\operatorname{obj} \mathbf{Set})^I$

  so objects are I-indexed families of sets

  $X = (X_i \mid i \in I)$

- morphisms $f : X \to Y$ in $\mathbf{Set}^I$ are I-indexed families of functions

  $f = (f_i \in \mathbf{Set}(X_i, Y_i) \mid i \in I)$

  composition in Set

- composition: $(g \circ f) = (g_i \circ f_i \mid i \in I)$

  identities: $\mathrm{id}_X = (\mathrm{id}_{X_i} \mid i \in I)$

  identity in Set

15.5

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

functor $F : \mathbf{Set}^I \longrightarrow \mathbf{Set}/I$

on objects: $F(X) \triangleq \left(\begin{array}{c} \{(i,x) \mid i \in I \;\&\; x \in X_i\} \\ \big\downarrow {\scriptstyle fst} \\ I \end{array} \right)$

on morphisms:

$$F(X \xrightarrow{f} X') \triangleq \begin{tikzcd} \left\{(i,x) \,\middle|\, \begin{array}{l} i \in I \;\& \\ x \in X_i \end{array}\right\} \arrow[rr] \arrow[dr] & & \left\{(i,x') \,\middle|\, \begin{array}{l} i \in I \;\& \\ x' \in X'_i \end{array}\right\} \arrow[dl] \\ & I & \end{tikzcd}$$

$$(i,x) \longmapsto (i, f_i\, x)$$

Example : $\mathbf{Set}^I \simeq \mathbf{Set}/I$

functor $G : \mathbf{Set}/I \to \mathbf{Set}^I$

on objects : $G\left(\begin{array}{c} E \\ \downarrow p \\ I \end{array}\right) \triangleq \left(\{e \in E \mid pe = i\} \mid i \in I\right)$

on morphisms :

$$G\left(\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & I & \end{tikzcd}\right) \triangleq GE \xrightarrow{Gf} GE'$$

where for each $i \in I$

$$(Gf)_i\, e \triangleq f(e)$$

Example : $\mathbf{Set}^I \simeq \mathbf{Set}/I$

There are natural isomorphisms

$$\eta : \mathrm{Id}_{\mathbf{Set}^I} \cong G \circ F$$

$$\varepsilon : F \circ G \cong \mathrm{Id}_{\mathbf{Set}/I}$$

defined as follows ...

Example : $\mathbf{Set}^I \simeq \mathbf{Set}/I$

$$\eta : \mathrm{Id}_{\mathbf{Set}^I} \cong G \circ F$$

for each $X \in \mathbf{Set}^I$ & $i \in I$, there is a bijection

$$(G(FX))_i = \{(i,x) \mid x \in X_i\} \overset{(\eta_X)_i}{\cong} X_i$$

$$(i,x) \longmapsto x$$

$$(i,x) \longleftarrow\!\shortmid\; x$$

(check ~~that~~ this is natural in X)

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

$$\varepsilon : F \circ G \cong \mathrm{Id}_{\mathbf{Set}/I}$$

For each $\begin{smallmatrix} E \\ \downarrow p \\ I \end{smallmatrix}$ in $\mathbf{Set}/I$

$$(i,e) \longmapsto e$$
$$(p(e), e) \longleftarrow\!\shortmid\; e$$

$$F\left(G\left(\begin{smallmatrix} E \\ \downarrow p \\ I \end{smallmatrix}\right)\right) = \begin{tikzcd} \{(i,e) \in I \times E \mid p(e) = i\} \arrow[d, "fst"'] \arrow[r, phantom, "\cong"] & E \arrow[dl, "p"] \\ I & \end{tikzcd}$$

(check this is natural in $\begin{smallmatrix} E \\ \downarrow p \\ I \end{smallmatrix}$)

<u>FACT</u> Given $p : I \to J$ in **Set**,

$$\mathbf{Set}/J \simeq \mathbf{Set}^J \xrightarrow{p^*} \mathbf{Set}^I \simeq \mathbf{Set}/I$$

is the functor "pullback along p"

Can generalize from **Set** to any category $\mathbb{C}$ with pullbacks & model Σ / Π types by left/right adjoints to pullback functor – see locally cartesian closed categories in literature.

Presheaf Categories

Let $\mathbb{C}$ be a small category.

The functor category $\boxed{\mathbf{Set}^{\mathbb{C}^{op}}}$

is called the **category of presheaves on** $\mathbb{C}$

- objects are contravariant functors from $\mathbb{C}$ to $\mathbf{Set}$
- morphisms are natural transformations

Much used at the moment to give semantics for various dependently-typed languages.

Yoneda functor

$$\boxed{y : \mathbb{C} \longrightarrow \mathbf{Set}^{\mathbb{C}^{op}}}$$

where $\mathbb{C}$ is a small category

is the Curried version of the hom functor

$$\mathbb{C} \times \mathbb{C}^{op} \cong \mathbb{C}^{op} \times \mathbb{C} \xrightarrow{H_{\mathbb{C}}} \mathbf{Set}$$

Yoneda functor

$$\boxed{y : \mathbb{C} \longrightarrow \mathbf{Set}^{\mathbb{C}^{op}}}$$

where $\mathbb{C}$ is a small category

is the curried version of the hom functor

$$\mathbb{C} \times \mathbb{C}^{op} \cong \mathbb{C}^{op} \times \mathbb{C} \xrightarrow{H_{\mathbb{C}}} \mathbf{Set}$$

- so for each $X \in \mathbb{C}$, $y(X) \in \mathbf{Set}^{\mathbb{C}^{op}}$ is the functor

$$\mathbb{C}(-, X) : \mathbb{C}^{op} \longrightarrow \mathbf{Set}$$

$$\begin{array}{c} Z \\ f \big\downarrow \\ Y \end{array} \text{ in } \mathbb{C} \longmapsto \begin{array}{ccc} \mathbb{C}(Z, X) & & g \circ f \\ \uparrow & & \overline{\uparrow} \\ \mathbb{C}(Y, X) & \ni & g \end{array}$$

15.14

Yoneda functor

$$\boxed{y : \mathbb{C} \longrightarrow \mathrm{Set}^{\mathbb{C}^{op}}}$$

where $\mathbb{C}$ is a small category

is the Curried version of the hom functor

$$\mathbb{C} \times \mathbb{C}^{op} \cong \mathbb{C}^{op} \times \mathbb{C} \xrightarrow{H_{\mathbb{C}}} \mathrm{Set}$$

- So for each $X \in \mathbb{C}$, $y(X) \in \mathrm{Set}^{\mathbb{C}^{op}}$ is the functor

$$\mathbb{C}(-, X) : \mathbb{C}^{op} \longrightarrow \mathrm{Set}$$

$$\begin{array}{c} Z \\ f \big\downarrow \\ Y \end{array} \text{ in } \mathbb{C} \quad \longmapsto \quad \begin{array}{c} \mathbb{C}(Z, X) \\ \big\uparrow \\ \mathbb{C}(Y, X) \end{array}$$

this function is often written as f^*

(N.B. $\mathbb{C}(-, X)$ is a functor)

Yoneda functor

$$\boxed{y : \mathbb{C} \longrightarrow \mathbf{Set}^{\mathbb{C}^{op}}}$$

where $\mathbb{C}$ is a small category

is the curried version of the hom functor

$$\mathbb{C} \times \mathbb{C}^{op} \cong \mathbb{C}^{op} \times \mathbb{C} \xrightarrow{H_{\mathbb{C}}} \mathbf{Set}$$

- so for each $Y \xrightarrow{f} X$ in $\mathbb{C}$, $y(Y) \xrightarrow{y(f)} y(X)$ in $\mathbf{Set}^{\mathbb{C}^{op}}$ is the natural transformation whose component at any $Z \in \mathbb{C}^{op}$ is the function

$$\begin{array}{ccc} y(Y)(Z) & \xrightarrow{\quad y(f)_Z \quad} & y(X)(Z) \\ \| & & \| \\ \mathbb{C}(Z, Y) & & \mathbb{C}(Z, X) \\ g & \longmapsto & f \circ g \end{array}$$

Yoneda functor

$$\boxed{y : \mathbb{C} \longrightarrow \mathbf{Set}^{\mathbb{C}^{op}}}$$

where $\mathbb{C}$ is a small category

is the Curried version of the hom functor

$$\mathbb{C} \times \mathbb{C}^{op} \cong \mathbb{C}^{op} \times \mathbb{C} \xrightarrow{H_{\mathbb{C}}} \mathbf{Set}$$

- So for each $Y \xrightarrow{f} X$ in $\mathbb{C}$, $y(Y) \xrightarrow{y(f)} y(X)$ in $\mathbf{Set}^{\mathbb{C}^{op}}$ is the natural transformation whose component at any $Z \in \mathbb{C}^{op}$ is the function

$$\begin{array}{ccc} y(Y)(Z) & \xrightarrow{\quad y(f)_Z \quad} & y(X)(Z) \\ \parallel & & \parallel \\ \mathbb{C}(Z, Y) & & \mathbb{C}(Z, X) \\ g & \longmapsto & f \circ g \end{array}$$

N.B. this does give a natural transformation

this function is often written as f_*

15.17

Yoneda Lemma

For each small category $\mathbb{C}$, each $X \in \mathbb{C}$ and each $F \in \mathbf{Set}^{\mathbb{C}^{op}}$, there is a bijection of sets

$$\eta_{X,F} : \underbrace{\mathbf{Set}^{\mathbb{C}^{op}}(y(X), F)}_{\text{the set of natural transformations from the functor } y(X) : \mathbb{C}^{op} \to \mathbf{Set} \text{ to the functor } F : \mathbb{C}^{op} \to \mathbf{Set}} \cong \underbrace{F(X)}_{\text{the value of } F : \mathbb{C}^{op} \to \mathbf{Set} \text{ at } X}$$

Yoneda Lemma

For each small category $\mathbb{C}$, each $X \in \mathbb{C}$ and each $F \in \mathbf{Set}^{\mathbb{C}^{op}}$, there is a bijection of sets

$$\eta_{X,F} : \underbrace{\mathbf{Set}^{\mathbb{C}^{op}}(y(X), F)}_{\text{the set of natural transformations from the functor } y(X):\mathbb{C}^{op}\to\mathbf{Set} \text{ to the functor } F:\mathbb{C}^{op}\to\mathbf{Set}} \cong \underbrace{F(X)}_{\text{the value of } F:\mathbb{C}^{op}\to\mathbf{Set} \text{ at } X}$$

which is natural in both X and F.