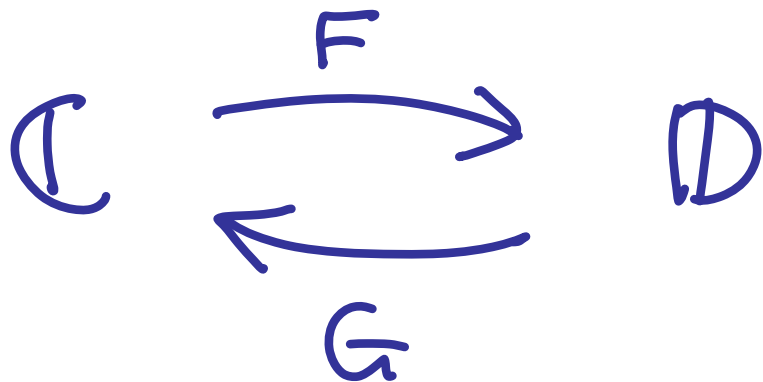


Equivalence

Two categories \mathbb{C} & \mathbb{D} are **isomorphic** if they are isomorphic objects in the category of categories (of some size), that is, there are

functors



satisfying

$$\begin{aligned} \text{Id}_{\mathbb{C}} &= G \circ F \\ F \circ G &= \text{Id}_{\mathbb{D}} \end{aligned}$$

(in which case, as usual, we write $\mathbb{C} \cong \mathbb{D}$)

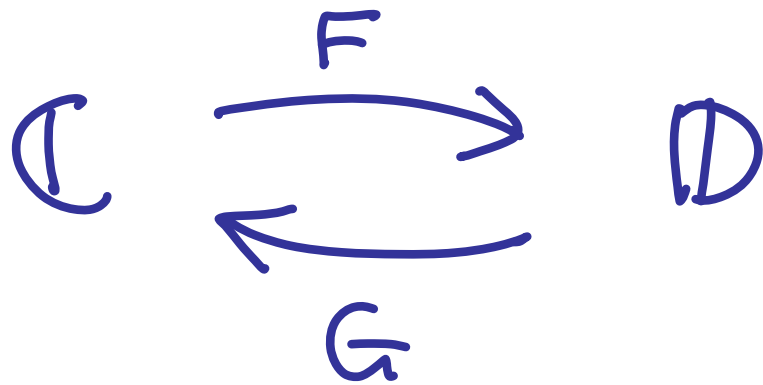
Equivalence

Two categories \mathbb{C} & \mathbb{D} are **equivalent** if there are

functors

&

natural isomorphisms



$$\eta: \text{Id}_{\mathbb{C}} \cong G \circ F$$

$$\varepsilon: F \circ G \cong \text{Id}_{\mathbb{D}}$$

in which case one writes

$$\mathbb{C} \cong \mathbb{D}$$

Some deep results in mathematics
take the form of equivalences:

E.g.

Stone duality: $(\text{Category of Boolean algebras})^{\text{op}} \simeq$ Category of compact, tot. disconn. Hausdorff spaces

Gelfand duality: $(\text{abelian } C^* \text{ algebras})^{\text{op}} \simeq$ Compact Hausdorff Spaces

Example: $\text{Set}^I \cong \text{Set}/I$

Set/I is a slice category [Ex. Sh.4, qu.6]

- objects are (x, f) where $f \in \text{Set}(x, I)$
- morphisms $g: (x, f) \rightarrow (x', f')$ are $g \in \text{Set}(x, x')$ satisfying $f' \circ g = f$ in Set
- composition & identities - as for Set

For each $I \in \text{obj Set}$, let $\boxed{\text{Set}^I}$ be the category with

- $\text{obj}(\text{Set}^I) \triangleq (\text{obj Set})^I$
so objects are I -indexed families of sets
 $X = (X_i \mid i \in I)$

- morphisms $f: X \rightarrow Y$ in Set^I are I -indexed families of functions
 $f = (f_i \in \text{Set}(X_i, Y_i) \mid i \in I)$

Composition in Set

- composition: $(g \circ f) = (g_i \circ f_i \mid i \in I)$
identities: $\text{id}_X = (\text{id}_{X_i} \mid i \in I)$

Identity in Set 15.5

Example: $\text{Set}^I \simeq \text{Set}/I$

functor $F: \text{Set}^I \rightarrow \text{Set}/I$

on objects: $F(X) \triangleq \left(\begin{array}{c} \{(i, x) \mid i \in I \ \& \ x \in X_i\} \\ \downarrow \text{fst} \\ I \end{array} \right)$

on morphisms:

$F(X \xrightarrow{f} X') \triangleq$

$$\begin{array}{ccc} \{(i, x) \mid i \in I \ \& \ x \in X_i\} & \longrightarrow & \{(i, x') \mid i \in I \ \& \ x' \in X'_i\} \\ & \searrow & \swarrow \\ & I & \\ (i, x) & \xrightarrow{\quad} & (i, f_i x) \end{array}$$

Example: $\text{Set}^I \cong \text{Set}/I$

functor $G: \text{Set}/I \rightarrow \text{Set}^I$

on objects: $G\left(\begin{array}{c} E \\ \downarrow p \\ I \end{array}\right) \triangleq \left(\{e \in E \mid pe = i\} \mid i \in I \right)$

on morphisms:

$G\left(\begin{array}{ccc} E & \xrightarrow{f} & E' \\ p \downarrow & & \downarrow p' \\ & I & \end{array}\right) \triangleq G E \xrightarrow{Gf} G E' \quad \text{where for}$
each $i \in I$
 $(Gf)_i e \triangleq f(e)$

Example: $\text{Set}^I \cong \text{Set}/I$

There are natural isomorphisms

$$\eta : \text{Id}_{\text{Set}^I} \cong G \circ F$$

$$\varepsilon : F \circ G \cong \text{Id}_{\text{Set}/I}$$

defined as follows...

Example: $\text{Set}^I \cong \text{Set}/I$

$$\eta: \text{Id}_{\text{Set}^I} \cong G \circ F$$

for each $x \in \text{Set}^I$ & $i \in I$, there is a bijection

$$(G(Fx))_i = \{(i, x) \mid x \in X_i\} \stackrel{(i, x)}{\cong} X_i$$

$$(i, x) \longmapsto x$$

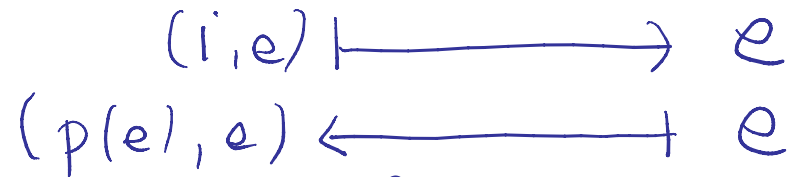
$$(i, x) \longleftarrow x$$

(check that this is natural in X)

Example: $\text{Set}^I \cong \text{Set}/I$

$$\varepsilon : F \circ G \cong \text{Id}_{\text{Set}/I}$$

For each $\begin{matrix} E \\ \downarrow \\ I \end{matrix}$ in Set/I



$$F\left(G\left(\begin{matrix} E \\ \downarrow \\ I \end{matrix}\right)\right) = \{(i, e) \in I \times E \mid p(e) = i\} \cong E$$

$\downarrow \text{fst}$
 I

$\swarrow p$
 E

(check this is natural in $\begin{matrix} E \\ \downarrow \\ I \end{matrix}$)

FACT Given $p: I \rightarrow J$ in Set ,

$$\text{Set}/J \simeq \text{Set}^J \xrightarrow{p^*} \text{Set}^I \simeq \text{Set}/I$$

is the functor "pullback along p "

Can generalize from Set to any
category \mathcal{C} with pullbacks & model
 Σ/Π types by left/right adjoints
to pullback functor - see locally cartesian
closed categories in literature.

Presheaf Categories

Let \mathbb{C} be a small category.

The functor category $\text{Set}^{\mathbb{C}^{\text{op}}}$

is called the **category of presheaves on \mathbb{C}**

- objects are contravariant functors from \mathbb{C} to Set
- morphisms are natural transformations

Much used at the moment to give semantics for various dependently-typed languages.

Yoneda functor

$$y : \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$$

where \mathcal{C} is a small category

is the Curried version of the hom functor

$$\mathcal{C} \times \mathcal{C}^{\text{op}} \cong \mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{H_{\mathcal{C}}} \text{Set}$$

Yoneda functor

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● So for each $x \in \mathcal{C}$, $y(x) \in \text{Set}^{\mathcal{C}^{\text{op}}}$ is the functor

$$\mathcal{C}(-, x) : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$$

$$\begin{array}{ccc} f \downarrow & z & \mathcal{C}(z, x) & g \circ f \\ & \text{in } \mathcal{C} & \uparrow & \uparrow \\ & & \mathcal{C}(y, x) & \cong \\ & & & g \end{array}$$

Yoneda functor

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$$\begin{array}{ccc} z & & \mathcal{C}(z, x) \\ f \downarrow \text{ in } \mathcal{C} & \mapsto & \uparrow \\ y & & \mathcal{C}(y, x) \end{array}$$

this function is often written as f^*

(N.B. $\mathcal{C}(-, x)$ is a functor)

Yoneda functor

$$y : \mathcal{C} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$$

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- So for each $Y \xrightarrow{f} X$ in \mathcal{C} , $y(Y) \xrightarrow{y(f)} y(X)$ in $\text{Set}^{\mathcal{C}^{\text{op}}}$ is the natural transformation whose component at any $Z \in \mathcal{C}^{\text{op}}$ is the function

$$\begin{array}{ccc}
 y(Y)(Z) & \xrightarrow{y(f)_Z} & y(X)(Z) \\
 \parallel & & \parallel \\
 \mathcal{C}(Z, Y) & & \mathcal{C}(Z, X) \\
 g & \xrightarrow{\quad} & f \circ g
 \end{array}$$

Yoneda functor

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 \mathcal{C}(Z, Y) & & \mathcal{C}(Z, X) \\
 g & \xrightarrow{\quad} & f \circ g
 \end{array}$$

NB. this does give a natural transformation

this function is often written as

f_* 15.17

Yoneda Lemma

For each small category \mathcal{C} , each $x \in \mathcal{C}$
and each $F \in \text{Set } \mathcal{C}^{\text{op}}$, there is a
bijection of sets

$$\eta_{x,F}: \text{Set } \mathcal{C}^{\text{op}}(y(x), F) \cong F(x)$$

the value of
 $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$
at x

the set of natural transformations
from the functor $y(x): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$
to the functor $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

Yoneda Lemma

For each small category \mathcal{C} , each $X \in \mathcal{C}$
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The value of
 $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$
at X

the set of natural transformations
from the functor $y(X): \mathcal{C}^{\text{op}} \rightarrow \text{Set}$
to the functor $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

which is natural in both X and F .