This lecture

Some category theory relevant to modelling type theories with

$\boxed{\text{dependent types}}$

Will restrict attention to what it looks like just in **Set** rather than in full generality

[See e.g.: M. Hofmann, "Syntax & Semantics of Dependent Types", pp 79-130 of A. Pitts & P. Dybjer, "Semantics & Logics of Computation (CUP, 1997)]

Simple types

$$x_1 : T_1, \dots, x_n : T_n \vdash t(x_1, \dots, x_n) : T$$

Simple types

$$x_1 : T_1, \ldots, x_n : T_n \vdash t(x_1, \ldots, x_n) : T$$

Dependent types

$$x_1 : T_1, \ldots, x_n : T_n \vdash t(x_1, \ldots, x_n) : T(x_1, \ldots, x_n)$$

& more generally

$$x_1 : T_1, x_2 : T_2(x_1), x_3 : T_3(x_1, x_2), \ldots \vdash t : T(x_1, x_2, \ldots)$$

If types denote sets,
then a dependent type

$$T(x) \qquad [\, x : T' \,]$$

should denote an
indexed family of sets

$$E = (\, E_i \mid i \in I \,)$$

i.e. E is a function $I \to \mathrm{obj}(\mathrm{Set})$

For each $I \in \operatorname{obj} \mathbf{Set}$, let $\boxed{\mathbf{Set}^I}$ be the category with

- $\operatorname{obj}(\mathbf{Set}^I) \triangleq (\operatorname{obj} \mathbf{Set})^I$
  so objects are I-indexed families of sets
  $X = (X_i \mid i \in I)$

- morphisms $f : X \to Y$ in $\mathbf{Set}^I$ are I-indexed families of functions
  $f = (f_i \in \mathbf{Set}(X_i, Y_i) \mid i \in I)$

composition in Set

- composition: $(g \circ f) = (g_i \circ f_i \mid i \in I)$
  identities: $\mathrm{id}_X = (\mathrm{id}_{X_i} \mid i \in I)$

identity in Set

For each $p : I \to J$ in $\mathbf{Set}$, let

$$\boxed{p^* : \mathbf{Set}^J \to \mathbf{Set}^I}$$

be the functor

$$p^*\left(\begin{array}{c} Y_j \\ \downarrow f_j \\ Y'_j \end{array} \middle|\, j \in J \right) \triangleq \left(\begin{array}{c} Y_{p(i)} \\ \downarrow f_{p(i)} \\ Y'_{p(i)} \end{array} \middle|\, i \in I \right)$$

i.e. p^* takes J-indexed families of sets/functions to I-indexed ones by precomposing with p

Dependent products
of families of sets

For $I, J \in \text{obj}\, Set$, projection $\pi_1 : I \times J \to I$ gives a functor $\pi_1^* : Set^I \to Set^{I \times J}$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\ \pi_1^* \text{ has a left adjoint} \\ \Sigma : Set^{I\times J} \to Set^I\end{array}}$$

<u>Proof</u> For each $E \in Set^{I \times J}$ we give

$$\Sigma E \in Set^I \quad \& \quad \eta_E : E \to \pi_1^*(\Sigma E)$$

with required universal property ...

For each $E \in \mathbf{Set}^{I \times J}$, we define $\Sigma E \in \mathbf{Set}^{I}$ by:

$$(\Sigma E)_i \triangleq \Sigma_{j \in J} E_{(i,j)} = \{(j,e) \mid j \in J \wedge e \in E_{(i,j)}\}$$

(all $i \in I$)

and $\eta_E \in \mathbf{Set}^{I \times J}(E, \pi_1^*(\Sigma E))$ by

$$(\eta_E)_{(i,j)} : E_{(i,j)} \to (\Sigma E)_i$$
$$e \mapsto (j,e)$$

(all $(i,j) \in I \times J$)

Universal property of $\eta_E : E \to \pi_1^*(\Sigma E)$:

Given any $X \in \mathbf{Set}^{I}$ & $f : E \to \pi_1^*(X)$ in $\mathbf{Set}^{I \times J}$, we have:

existence

$$
\begin{tikzcd}
E \arrow[r, "\eta_E"] \arrow[dr, "f"'] & \pi_1^*(\Sigma E) \arrow[d, "\pi_1^*(\hat{f})"] \\
& \pi_1^*(X)
\end{tikzcd}
\qquad
\boxed{
\begin{tikzcd}
\Sigma E \arrow[d, "\hat{f}"] \\
X
\end{tikzcd}
}
\leftarrow \text{in } \mathbf{Set}^{I}
$$

where $\boxed{\hat{f}_i(j, e) \triangleq f_{(i,j)}(e) \quad \left(\text{all } \begin{array}{l} i \in I \\ j \in J \\ e \in E_{(i,j)} \end{array}\right)}$

Universal property of $\eta_E : E \to \pi_1^*(\Sigma E)$

Given any $X \in Set^I$ & $f : E \to \pi_1^*(X)$ in $Set^{I \times J}$, we have:

uniqueness

if

$$\begin{tikzcd} E \arrow[r, "\eta_E"] \arrow[dr, "f"'] & \pi_1^*(\Sigma E) \arrow[d, "\pi_1^*(g)"] \\ & \pi_1^*(X) \end{tikzcd} \qquad \boxed{\begin{tikzcd} \Sigma E \arrow[d, "g"] \\ X \end{tikzcd}} \leftarrow \text{in } Set^I$$

then for all $i \in I$, $(j,e) \in (\Sigma E)_i$, have

$\hat{f}_i(j,e) = f_{(i,j)}(e) = (\pi_1^* g \circ \eta_E)_{(i,j)}\, e = g_i(j,e)$ So $g = \hat{f}$ □

14.10

Dependent functions

for families of sets

For $I, J \in \mathrm{obj}\,\mathbf{Set}$, projection $\pi_1 : I \times J \to I$ gives a functor $\pi_1^* : \mathbf{Set}^I \longrightarrow \mathbf{Set}^{I \times J}$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\ \pi_1^* \text{ has a right adjoint} \\ \quad \Pi : \mathbf{Set}^{I \times J} \longrightarrow \mathbf{Set}^I\end{array}}$$

<u>Proof</u> For each $E \in \mathbf{Set}^{I \times J}$ we give

$\Pi E \in \mathbf{Set}^I$ & $\varepsilon_E : \pi_1^*(\Pi E) \to E$

with required universal property ...

For each $E \in \mathbf{Set}^{I \times J}$, we define $\Pi E \in \mathbf{Set}^{I}$ by:

$$(\Pi E)_i \triangleq \prod_{j \in J} E_{(i,j)}$$
$$= \{ f \subseteq (\Sigma E)_i \mid f \text{ is single-valued } \& \text{ total} \}$$

where $f \subseteq (\Sigma E)_i$ is

single-valued if $(\forall j \in J)(\forall e, e' \in E_{(i,j)})\ (j,e) \in f \wedge (j,e') \in f \Rightarrow e = e'$

total if $(\forall j \in J)(\exists e \in E_{(i,j)})\ (j,e) \in f$

ie. each $f \in (\Pi E)_i$ is a <u>dependently typed</u> function mapping elements $j \in J$ to elements of $E_{(i,j)}$ ← depends on argument j

For each $E \in \mathbf{Set}^{I\times J}$, we define $\Pi E \in \mathbf{Set}^{I}$ by:

$$(\Pi E)_i \triangleq \Pi_{j\in J} E_{(i,j)}$$
$$= \{ f \subseteq (\Sigma E)_i \mid f \text{ is single-valued \& total} \}$$

and $\varepsilon_E \in \mathbf{Set}^{I\times J}(\pi_1^*(\Pi E), E)$ by

$$(\varepsilon_E)_{(i,j)} : (\Pi E)_i \to E_{(i,j)}$$
$$f \mapsto f(j)$$

(all $(i,j) \in I \times J$)

the unique $e \in E_{(i,j)}$ such that $(j,e) \in f$

Universal property of $\varepsilon_E : \pi_1^*(\Pi E) \to E$

Given any $X \in \mathbf{Set}^I$ and $f : \pi_1^*(X) \to E$ in $\mathbf{Set}^{I \times J}$, we have:

existence

$$\begin{tikzcd} \Pi E \\ X \arrow[u, "\hat{f}"] \end{tikzcd} \qquad \begin{tikzcd} \pi_1^*(\Pi E) \arrow[r, "\varepsilon_E"] & E \\ \pi_1^*(X) \arrow[u, "\pi_1^*(\hat{f})"] \arrow[ur, "f"'] & \end{tikzcd}$$

in $\mathbf{Set}^I$

where $\hat{f}_i(x) \triangleq \{(j, f_{(i,j)}(x)) \mid j \in J\}$

(all $i \in I$, $x \in X_i$)

Universal property of $\varepsilon_E : \pi_1^*(\Pi E) \to E$

Given any $X \in \mathbf{Set}^I$ and $f : \pi_1^*(X) \to E$ in $\mathbf{Set}^{I \times J}$, we have :

uniqueness

if

$$\boxed{\begin{tikzcd} \Pi E \\ X \arrow[u, "g"] \end{tikzcd}}$$

in Set^I

$$\begin{tikzcd} \pi_1^*(\Pi E) \arrow[r, "\varepsilon_E"] & E \\ \pi_1^*(X) \arrow[u, "\pi_1^*(g)"] \arrow[ur, "f"'] & \end{tikzcd}$$

then $\hat{f}_i\, x\, j = f_{(i,j)}\, x = (\varepsilon_E \circ \pi_1^* g)_{(i,j)}\, x = g_i\, x\, j$

so $g = \hat{f}$ $\square$