

Given $\mathbb{C} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \mathbb{D}$,

if there is some $\theta : H_{\mathbb{D}}^{\circ}(F \times \text{id}) \cong H_{\mathbb{C}}^{\circ}(\text{id} \times G)$
one says

F is a left adjoint for G

G is a right adjoint for F

and writes

$F \dashv G$

The existence of θ is sometimes indicated by writing

$$\theta \curvearrowright \frac{FX \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY} \curvearrowleft \theta^{-1}$$

Using this notation, can split the naturality condition for θ into two:

$$\frac{FX' \xrightarrow{Fu} FX \xrightarrow{g} Y}{X' \xrightarrow{u} X \xrightarrow{\bar{g}} GY}$$

$$\frac{FX \xrightarrow{g} Y \xrightarrow{v} Y'}{X \xrightarrow{\bar{g}} GY \xrightarrow{Gv} GY'}$$

Proposition. \mathcal{C} has binary products if & only if the diagonal functor $\Delta = \langle \text{id}, \text{id} \rangle : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ has a right adjoint.

Proposition A cartesian category \mathcal{C} has all exponentials if & only if for all $X \in \text{Obj } \mathcal{C}$, the functor $(-) \times X : \mathcal{C} \rightarrow \mathcal{C}$ has a right adjoint.

Proposition. \mathcal{C} has binary products if & only if the diagonal functor $\Delta = \langle \text{id}, \text{id} \rangle : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ has a right adjoint.

Both instances of the following theorem

- a very useful characterisation of when a functor has a right adjoint (or dually, has a left adjoint)

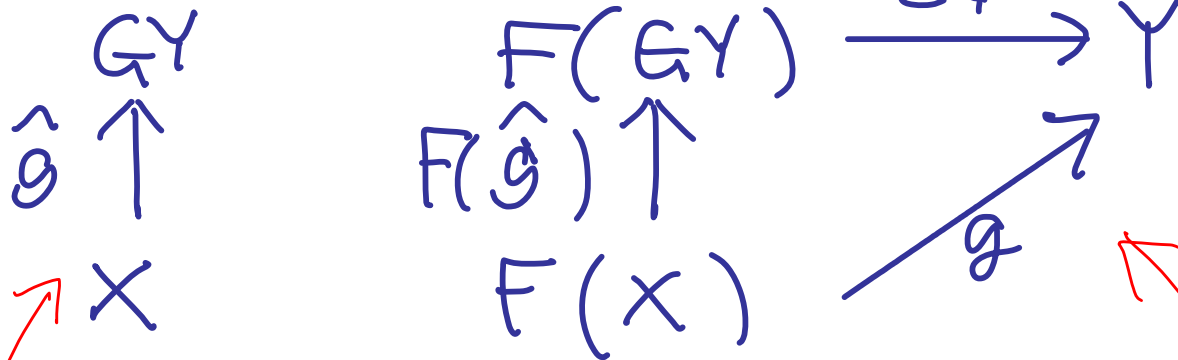
Proposition A cartesian category \mathcal{C} has all exponentials if & only if for all $X \in \text{Obj } \mathcal{C}$, the functor $(-) \times X : \mathcal{C} \rightarrow \mathcal{C}$ has a right adjoint.

Theorem

$F: \mathbb{C} \rightarrow \mathbb{D}$ has a right adjoint if & only if
 for all $Y \in \text{obj } \mathbb{D}$ there are
 $G_Y \in \text{obj } \mathbb{C}$ & $\epsilon_Y \in \mathbb{D}(F(G_Y), Y)$

with the universal property:

For all $X \in \text{obj } \mathbb{C}$ & $g \in \mathbb{D}(F(X), Y)$ there is
 a unique $\hat{g} \in \mathbb{C}(X, G_Y)$ satisfying $\epsilon_Y \circ F(\hat{g}) = g$



UP

in \mathbb{C}

in \mathbb{D}

Proof of the theorem - "only if" part:

given an adjunction (F, G, θ) , for each $Y \in \text{obj } \mathbb{D}$ produce $\epsilon_Y : F(GY) \rightarrow Y$ satisfying **UP**.

We have $\theta_{X,Y} : \mathbb{D}(FX, Y) \cong \mathbb{C}(X, GY)$ natural in X & Y

Define: $\epsilon_Y \triangleq \theta_{GY, Y}^{-1}(\text{id}_{GY}) : F(GY) \rightarrow Y$

In other words $\epsilon_Y = \overline{\text{id}_{GY}}$, i.e.
$$\frac{F(GY) \xrightarrow{\epsilon_Y} Y}{GY \xrightarrow{\text{id}} GY}$$

Given any $\begin{cases} g: FX \rightarrow Y & \text{in } \mathbb{D} \\ f: X \rightarrow GY & \text{in } \mathbb{C} \end{cases}$

by naturality we have

$$\frac{g: FX \rightarrow Y}{\bar{g}: X \rightarrow GY} \quad \& \quad \frac{\varepsilon_Y \circ Ff: FX \xrightarrow{Ff} F(GY) \xrightarrow{\overline{id_{GY}}} Y}{f: X \xrightarrow{f} GY \xrightarrow{id_{GY}} GY}$$

So $g = \varepsilon_Y \circ F(\bar{g})$

and $g = \varepsilon_Y \circ Ff \implies \bar{g} = f$

Thus we do have UP (with $\hat{g} \triangleq \bar{g}$).

Proof of the theorem - "if" part :

We are given $F: \mathbb{C} \rightarrow \mathbb{D}$ and for each $Y \in \text{Obj } \mathbb{D}$ a \mathbb{C} -object GY + \mathbb{C} -morphism $\epsilon_Y: F(GY) \rightarrow Y$ satisfying **UP**. We have to

① extend $Y \mapsto GY$ to a functor $G: \mathbb{D} \rightarrow \mathbb{C}$

② construct a natural iso $\theta: H_{\mathbb{D}}^0(F \text{ id}) \cong H_{\mathbb{C}}^0(\text{id} \times G)$

① For each \mathbb{D} -morphism $g: Y' \rightarrow Y$
we get $F(GY') \xrightarrow{\varepsilon_{Y'}} Y' \xrightarrow{g} Y$ and can
apply **UP** to get

$$Gg \triangleq (g \circ \varepsilon_{Y'})^\wedge : GY' \rightarrow GY$$

The uniqueness part of **UP** implies

$$G(\text{id}) = \text{id} \quad G(g' \circ g) = (Gg') \circ (Gg)$$

so we get a functor $G: \mathbb{D} \rightarrow \mathbb{C}$.



② Since for all $g: FX \rightarrow Y$, there is a unique $f: X \rightarrow GY$ with $g = \varepsilon_Y \circ Ff$,

$f \mapsto \bar{f} \triangleq \varepsilon_Y \circ Ff$ determines a

bijection $\mathcal{C}(X, GY) \cong \mathcal{D}(FX, Y)$

and it is natural in X & Y since

$$\underline{Gv \circ f \circ u = \varepsilon_Y \circ F(Gv \circ f \circ u)}$$

$$= (\varepsilon_Y \circ FGv) \circ Ff \circ Fu$$

by defⁿ
of Gv

$$\rightarrow = (v \circ \varepsilon_Y) \circ f \circ Fu$$

$$= v \circ \bar{f} \circ Fu$$

by defⁿ
of \bar{f} 13.20

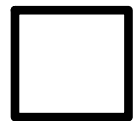
② Since for all $g: FX \rightarrow Y$, there is a unique $f: X \rightarrow GX$ with $g = \varepsilon_Y \circ Ff$,

$f \mapsto \bar{f} \triangleq \varepsilon_Y \circ Ff$ determines a

bijection $\mathcal{C}(X, GX) \cong \mathcal{D}(FX, Y)$

and it is natural in X & Y since ...

So we take θ to be the inverse of this natural isomorphism.



Dual of the theorem

$G: \mathbb{C} \leftarrow \mathbb{D}$ has a **left** adjoint if & only if
for all $X \in \text{obj } \mathbb{C}$ there are

$$FX \in \text{obj } \mathbb{D} \text{ \& } \eta_X \in \mathbb{C}(X, G(FX))$$

with the universal property:

For all $Y \in \text{obj } \mathbb{D}$ & $f \in \mathbb{C}(X, G(Y))$ there is
a unique $\hat{f} \in \mathbb{D}(FX, Y)$ satisfying $G(\hat{f}) \circ \eta_X = f$

$$\begin{array}{ccc} & G(FX) & \xleftarrow{\eta_X} X \\ & \downarrow G(\hat{f}) & \swarrow f \\ \hat{f} & & G(Y) \\ \downarrow & & \\ FX & & \\ \downarrow & & \\ Y & & \end{array}$$

up

in \mathbb{D}

in \mathbb{C}

E.g. from the dual version of the theorem we can conclude that the forgetful functor

$$U : \text{Mon} \rightarrow \text{Set}$$

has a left adjoint $F : \text{Set} \rightarrow \text{Mon}$,

because of the universal property of

$$F(\Sigma) = (\text{List}(\Sigma), e, \text{nil}) \quad \& \quad i_{\Sigma} : \Sigma \rightarrow \text{List}(\Sigma)$$

from Lecture 3.

$$U(F\Sigma)$$

Why are adjoint functors important / useful?

- UP usually embodies some useful mathematical construction (eg. "freely generated structures are left adjoints for forgetting structure") and pins it down uniquely up to iso