Given $\mathbb{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{D}$,

if there is some $\theta : H_{\mathbb{D}} \circ (F^{op} \times id) \cong H_{\mathbb{C}} \circ (id \times G)$

one says

F is a **left adjoint** for G

G is a **right adjoint** for F

and writes $\boxed{F \dashv G}$

The existence of θ is sometimes indicated by writing

$$\theta \circlearrowright \boxed{\dfrac{Fx \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY}} \circlearrowleft \theta^{-1}$$

Using this notation, can split the naturality condition for θ into two:

$$\frac{Fx' \xrightarrow{Fu} Fx \xrightarrow{g} Y}{X' \xrightarrow{u} X \xrightarrow{\bar{g}} GY} \qquad \frac{Fx \xrightarrow{g} Y \xrightarrow{v} Y'}{X \xrightarrow{\bar{g}} GY \xrightarrow{Gv} GY'}$$

Proposition. $\mathbb{C}$ has binary products if & only if the diagonal functor

$$\Delta = \langle id, id \rangle : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$$

has a right adjoint.

Proposition A cartesian category $\mathbb{C}$ has all exponentials if & only if for all $X \in \mathrm{Obj}\,\mathbb{C}$, the functor

$$(-) \times X : \mathbb{C} \to \mathbb{C}$$

has a right adjoint.

Proposition. $\mathbb{C}$ has binary products if & only if the diagonal functor

$$\Delta = \langle id, id\rangle : \mathbb{C} \to \mathbb{C}\times\mathbb{C}$$

has a right adjoint.

Both instances of the following theorem – a very useful characterisation of when a functor has a right adjoint (or dually, has a left adjoint)

Proposition A cartesian category $\mathbb{C}$ has all exponentials if & only if for all $X \in \mathrm{Obj}\,\mathbb{C}$, the functor

$$(-)\times X : \mathbb{C} \to \mathbb{C}$$

has a right adjoint.

Theorem

$F : \mathbb{C} \to \mathbb{D}$ has a right adjoint if & only if for all $Y \in \operatorname{obj}\mathbb{D}$ there are

$$GY \in \operatorname{obj}\mathbb{C} \quad \& \quad \varepsilon_Y \in \mathbb{D}(F(GY), Y)$$

with the universal property:

For all $X \in \operatorname{obj}\mathbb{C}$ & $g \in \mathbb{D}(F(X), Y)$ there is a unique $\hat{g} \in \mathbb{C}(X, GY)$ satisfying $\varepsilon_Y \circ F(\hat{g}) = g$

$$\begin{tikzcd} GY \\ X \arrow[u, "\hat{g}"] \end{tikzcd} \qquad \begin{tikzcd} F(GY) \arrow[r, "\varepsilon_Y"] & Y \\ F(X) \arrow[u, "F(\hat{g})"] \arrow[ur, "g"'] & \end{tikzcd}$$

in $\mathbb{C}$ in $\mathbb{D}$

$\boxed{\text{UP}}$

Proof of the theorem - "only if" part:

given an adjunction (F, G, θ), for each $Y \in \mathrm{Obj}\,\mathbb{D}$ produce $\varepsilon_Y : F(GY) \to Y$ satisfying $\boxed{\text{UP}}$.

We have $\theta_{X,Y} : \mathbb{D}(FX, Y) \cong \mathbb{C}(X, GY)$ natural in X & Y

Define : $\boxed{\varepsilon_Y \triangleq \theta^{-1}_{GY,Y}(id_{GY}) : F(GY) \to Y}$

In other words $\varepsilon_Y = \overline{id_{GY}}$, i.e.
$$\frac{F(GY) \xrightarrow{\varepsilon_Y} Y}{GY \xrightarrow{id} GY}$$

Given any $\begin{cases} g : FX \to Y & \text{in } \mathbb{D} \\ f : X \to GY & \text{in } \mathbb{C} \end{cases}$

by naturality we have

$$\frac{g : FX \to Y}{\bar{g} : X \to GY} \quad \& \quad \frac{\varepsilon_Y \circ Ff : FX \xrightarrow{Ff} F(GY) \xrightarrow{\overline{id_{GY}}} Y}{f \quad : X \xrightarrow[f]{} GY \xrightarrow[id_{GY}]{} GY}$$

So $\quad g = \varepsilon_Y \circ F(\bar{g})$

and $\quad g = \varepsilon_Y \circ Ff \implies \bar{g} = f$

Thus we do have $\boxed{\text{UP}}$ (with $\hat{g} \triangleq \bar{g}$).

Proof of the theorem – "if" part :

We are given $F : \mathbb{C} \to \mathbb{D}$ and for each $Y \in \mathrm{Obj}\,\mathbb{D}$ a $\mathbb{C}$-object GY + $\mathbb{C}$-morphism $\varepsilon_Y : F(GY) \to Y$ satisfying $\boxed{\text{UP}}$. We have to

① extend $Y \mapsto GY$ to a functor $G : \mathbb{D} \to \mathbb{C}$

② construct a natural iso $\theta : H_{\mathbb{D}} \circ (F \times \mathrm{id}) \cong H_{\mathbb{C}} \circ (\mathrm{id} \times G)$

① For each $\mathbb{D}$-morphism $g : Y' \to Y$ we get $F(GY') \xrightarrow{\varepsilon_{Y'}} Y' \xrightarrow{g} Y$ and can apply $\boxed{\text{UP}}$ to get

$$Gg \triangleq (g \circ \varepsilon_{Y'})^{\wedge} : GY' \longrightarrow GY$$

The uniqueness part of $\boxed{\text{UP}}$ implies

$$G(id) = id \qquad G(g' \circ g) = (Gg') \circ (Gg)$$

so we get a functor $G : \mathbb{D} \to \mathbb{C}$.

$\square$

② Since for all $g: FX \to Y$, there is a unique $f: X \to GY$ with $g = \varepsilon_Y \circ Ff$,

$f \mapsto \bar{f} \triangleq \varepsilon_Y \circ Ff$ determines a bijection $\mathbb{C}(X, GY) \cong \mathbb{D}(FX, Y)$

and it is natural in X & Y since

$$
\begin{aligned}
\overline{Gv \circ f \circ u} &= \varepsilon_{Y'} \circ F(Gv \circ f \circ u) \\
&= (\varepsilon_{Y'} \circ FGv) \circ Ff \circ Fu \\
&= (v \circ \varepsilon_Y) \circ Ff \circ Fu \\
&= v \circ \bar{f} \circ Fu
\end{aligned}
$$

by def^n of Gv

by def^n of $\bar{f}$

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2 Since for all $g : FX \to Y$, there is a unique $f : X \to GY$ with $g = \varepsilon_Y \circ Ff$,

$$f \longmapsto \bar{f} \triangleq \varepsilon_Y \circ Ff$$

determines a bijection $\mathbb{C}(X, GY) \cong \mathbb{D}(FX, Y)$ and it is natural in X & Y since ...

So we take θ to be the inverse of this natural isomorphism.

$\square$

Dual of the theorem

$G : \mathbb{C} \leftarrow \mathbb{D}$ has a left adjoint if & only if for all $X \in \mathrm{obj}\,\mathbb{C}$ there are

$$FX \in \mathrm{obj}\,\mathbb{D} \quad \& \quad \eta_X \in \mathbb{C}(X, G(FX))$$

with the universal property:

For all $Y \in \mathrm{obj}\,\mathbb{D}$ & $f \in \mathbb{C}(X, G(Y))$ there is a unique $\hat{f} \in \mathbb{D}(FX, Y)$ satisfying $G(\hat{f}) \circ \eta_X = f$

$$\begin{tikzcd} FX \arrow[d, "\hat{f}"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} G(FX) \arrow[d, "G(\hat{f})"'] & X \arrow[l, "\eta_X"'] \arrow[dl, "f"] \\ G(Y) & \end{tikzcd}$$

$\boxed{\text{UP}}$

in $\mathbb{D}$ 　　　　 in $\mathbb{C}$

E.g. from the dual version of the theorem we can conclude ~~that~~ the forgetful functor

$$U : \mathbf{Mon} \to \mathbf{Set}$$

has a left adjoint $F : \mathbf{Set} \to \mathbf{Mon}$, because of the universal property of

$$F(\Sigma) = (\mathrm{List}(\Sigma), @, nil) \quad \& \quad i_\Sigma : \Sigma \to \underset{\substack{\shortparallel \\ U(F\Sigma)}}{\mathrm{List}(\Sigma)}$$

from Lecture 3.

Why are adjoint functors important / useful ?

- $\boxed{\text{UP}}$ usually embodies some useful mathematical construction (eg. "freely generated structures are left adjoints for forgetting structure") and pins it down uniquely up to iso