

# Adjoint functors

Categories, functors & natural transformations were invented (by Eilenberg & MacLane) in order to formalize "adjoint situations".

They appear everywhere in mathematics, logic and (hence) CS.

Examples that we have seen ...

# Binary product in $\mathcal{C}$

$$(z, z) \rightarrow (x, y)$$

morphisms  
in  $\mathcal{C} \times \mathcal{C}$

$$z \rightarrow x \times y$$

morphisms in  $\mathcal{C}$

bijection correspondence :

$$\mathcal{C} \times \mathcal{C} \left( (z, z), (x, y) \right) \cong \mathcal{C} (z, x \times y)$$

$$(f, g) \longmapsto \langle f, g \rangle$$

$$(\pi_1 \circ h, \pi_2 \circ h) \longleftarrow h$$

furthermore, this bijection "is natural in  $x, y, z$ "  
(to be explained) 12.2

# Exponentials in $\mathcal{C}$

$$\underline{\underline{Z \times X \longrightarrow Y}}$$

$$Z \longrightarrow Y^X$$

morphisms in  $\mathcal{C}$

morphisms in  $\mathcal{C}$

bijective correspondence

$$\mathcal{C}(Z \times X, Y) \cong \mathcal{C}(Z, Y^X)$$

$$f \longmapsto \text{cur } f$$

$$\text{app} \circ (g \times \text{id}_X) \longleftarrow g$$

natural in  $X, Y, Z$

# Free monoids

$$\underline{\underline{\Sigma \rightarrow U(M, \cdot, e)}} \quad \text{in Set}$$

$$F\Sigma \rightarrow (M, \cdot, e) \quad \text{in Mon}$$

free monoid on set  $\Sigma$   
(List( $\Sigma$ ),  $-@-$ , nil)

bijjective correspondence

$$\text{Set}(\Sigma, U(M)) \cong \text{Mon}(F\Sigma, M)$$

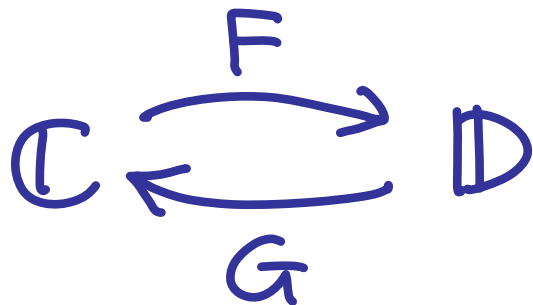
$$g \circ i_\Sigma \quad \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array}$$

natural in  $\Sigma$  &  $M$

# Adjunction

Definition An **adjunction** between two categories  $\mathbb{C}$  &  $\mathbb{D}$  is specified by

- functors



- bijections

$$\theta_{x,y} : \mathbb{D}(F(x), y) \cong \mathbb{C}(x, G(y))$$

for each  $x \in \text{Obj } \mathbb{C}$  &  $y \in \text{Obj } \mathbb{D}$

which are **natural in  $x$  &  $y$** , meaning...

For  $\theta_{X,Y} : \mathbb{D}(F(X), Y) \cong \mathbb{C}(X, G(Y))$

to be "natural in  $X$  &  $Y$ " means

for all  $\begin{cases} u : X' \rightarrow X \text{ in } \mathbb{C} \\ v : Y \rightarrow Y' \text{ in } \mathbb{D} \end{cases}$

and all  $g : F(X) \rightarrow Y$  in  $\mathbb{D}$

$$X' \xrightarrow{u} X \xrightarrow{\theta_{X,Y}(g)} G(Y) \xrightarrow{Gv} G(Y')$$

$$= \theta_{X',Y'} (F(X') \xrightarrow{Fu} F(X) \xrightarrow{g} Y \xrightarrow{v} Y')$$

For  $\theta_{XY} : \mathbb{D}(F(X), Y) \cong \mathbb{C}(X, G(Y))$

to be "natural in  $X$  &  $Y$ " means

for all  $\begin{cases} u : X' \rightarrow X \text{ in } \mathbb{C} \\ v : Y \rightarrow Y' \text{ in } \mathbb{D} \end{cases}$

and all  $g : F(X) \rightarrow Y$  in  $\mathbb{D}$

$$\begin{aligned} X' &\xrightarrow{u} X \xrightarrow{\theta_{XY}(g)} G(Y) \xrightarrow{Gv} G(Y') \\ &= \theta_{X'Y'} \quad F(X') \xrightarrow{Fu} F(X) \xrightarrow{g} Y \xrightarrow{v} Y' \end{aligned}$$

what has this  
to do with  
the concept of  
natural  
transformation?

# Hom functors

If  $\mathcal{C}$  is locally small, then we get a functor

$$H_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set}$$

with  $H_{\mathcal{C}}(X, Y) \triangleq \mathcal{C}(X, Y)$  and

$$H_{\mathcal{C}}\left(\begin{array}{c} (X, Y) \\ \xrightarrow{(f, g)} \\ (X', Y') \end{array}\right) \triangleq \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X', Y')$$

$h \mapsto g \circ h \circ f$

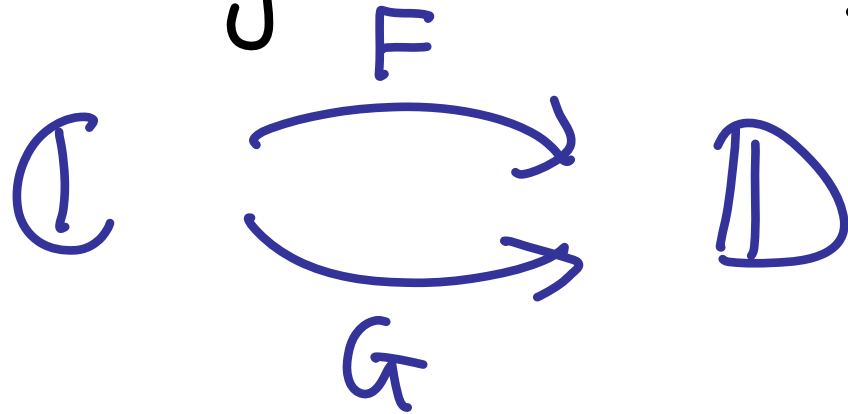
$$f: X' \rightarrow X \\ \text{in } \mathcal{C}$$

$$g: Y \rightarrow Y' \\ \text{in } \mathcal{C}$$



# Natural isomorphisms

Given categories and functors

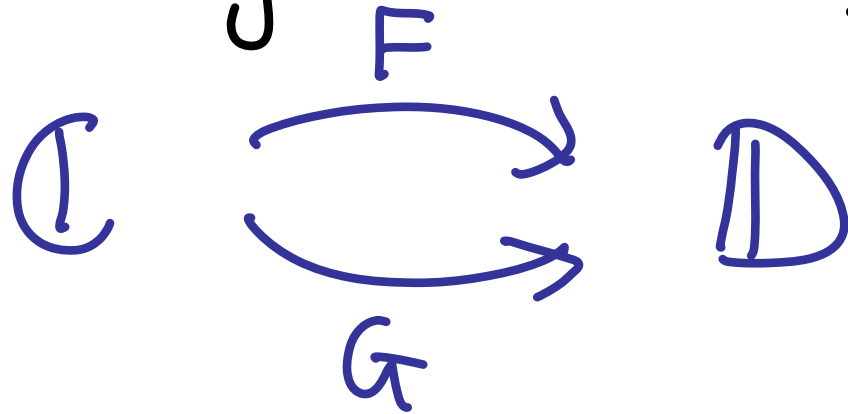


a natural isomorphism  $\theta: F \cong G$

is simply an isomorphism between  $F$  &  $G$  in the functor category  $\mathbb{D}^{\mathbb{C}}$ .

# Natural isomorphisms

Given categories and functors



FACT If  $\theta : F \rightarrow G$  is a nat. transf.<sup>n</sup>

and for each  $x \in \text{Obj } \mathbb{C}$ ,  $\theta_x : F(x) \rightarrow G(x)$  is an isomorphism in  $\mathbb{D}$ , then

$\theta_x^{-1} : G(x) \rightarrow F(x)$  gives a nat. transf.<sup>n</sup>

$\theta^{-1} : G \rightarrow F$  &  $F \cong G$  in  $\mathbb{D}^{\mathbb{C}}$

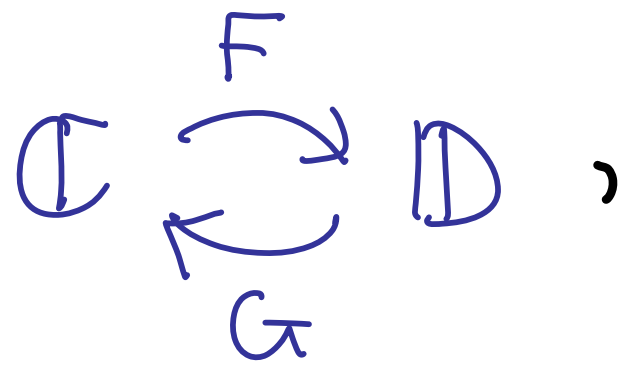
Given locally small categories  $\mathcal{C}$  &  $\mathcal{D}$ ,  
 if we have  $\mathcal{C} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \mathcal{D}$  we get  
 functors

$$\begin{array}{ccc}
 & \mathcal{C}^{\text{op}} \times \mathcal{D} & \\
 & \begin{matrix} \xrightarrow{F^{\text{op}} \times \text{id}} \\ \xrightarrow{\text{id} \times G} \end{matrix} & \mathcal{D}^{\text{op}} \times \mathcal{D} \\
 & & \begin{matrix} \xrightarrow{H_{\mathcal{D}}} \\ \xrightarrow{H_{\mathcal{C}}} \end{matrix} \\
 & & \text{Set}
 \end{array}$$

An adjunction  $(F, G, \theta)$  is give by a  
 nat. iso  $\theta : H_{\mathcal{D}}^{\text{op}}(F^{\text{op}} \times \text{id}) \cong H_{\mathcal{C}}^{\text{op}}(\text{id} \times G)$

# Terminology

Given



if there is some  $\theta : H_D^{\circ}(F \times \text{id}) \cong H_C^{\circ}(\text{id} \times G)$   
one says

$F$  is a **left adjoint** for  $G$

$G$  is a **right adjoint** for  $F$

and writes

$$F \dashv G$$

Notation associated with an adjunction  
(F, G,  $\theta$ )

Given  $\begin{cases} g: FX \rightarrow Y \\ f: X \rightarrow GX \end{cases}$

we write  $\begin{cases} \bar{g} \triangleq \theta_{X,Y}(g) : X \rightarrow GX \\ \bar{f} \triangleq \theta_{X,Y}^{-1}(f) : FX \rightarrow Y \end{cases}$

Thus  $\bar{g} = g$ ,  $\bar{f} = f$  and naturality means

$$\underline{v \circ g \circ fu = Gv \circ \bar{g} \circ u}$$

The existence of  $\theta$  is sometimes indicated by writing

$$\frac{FX \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GX}$$

The existence of  $\theta$  is sometimes indicated by writing

$$\theta \curvearrowright \frac{FX \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY} \curvearrowleft \theta^{-1}$$

Using this notation, can split the naturality condition for  $\theta$  into two:

$$\frac{FX' \xrightarrow{Fu} FX \xrightarrow{g} Y}{X' \xrightarrow{u} X \xrightarrow{\bar{g}} GY} \qquad \frac{FX \xrightarrow{g} Y \xrightarrow{v} Y'}{X \xrightarrow{\bar{g}} GY \xrightarrow{Gv} GY'}$$