The category of small categories, $\mathbf{Cat}$

- objects are all small categories
- morphisms $\mathbf{Cat}(\mathbb{C},\mathbb{D})$ are all functors $F:\mathbb{C}\to\mathbb{D}$
- composition & identities – for functors, as before

Cat has a terminal object

- Terminal object in **Cat** is

$$\boxed{\begin{tikzcd} \ast \arrow[loop above, "id_\ast"] \end{tikzcd}} = \text{one-object, one morphism category}$$

Cat has binary products

- Binary product $\mathbb{C} \xleftarrow{\pi_1} \mathbb{C}\times\mathbb{D} \xrightarrow{\pi_2} \mathbb{D}$
  - objects of $\mathbb{C}\times\mathbb{D}$ are pairs (X,Y) with $X \in \mathrm{obj}\,\mathbb{C}$ & $Y \in \mathrm{obj}\,\mathbb{D}$
  - morphisms $(X,Y) \to (X',Y')$ are pairs (f,g) of morphisms $f \in \mathbb{C}(X,X')$, $g \in \mathbb{D}(Y,Y')$
  - composition & identities as in $\mathbb{C}$ & $\mathbb{D}$
  - $\pi_1(X,Y) = X \qquad \pi_1(f,g) = f$
    $\pi_2(X,Y) = Y \qquad \pi_2(f,g) = g$

Cat not only has finite products, it is also cartesian closed – exponentials in **Cat** are called **functor categories** and to define them we need to consider **natural transformations** which are the appropriate notion of morphism between functors.

Natural Transformations

Motivating example: fix a set $S \in \mathrm{Obj}\,\mathbf{Set}$ and consider the two functors $F, G : \mathbf{Set} \to \mathbf{Set}$ given by

$$\begin{cases} F(X) \triangleq S \times X \\ F(f) \triangleq \mathrm{id}_S \times f \end{cases} \qquad \begin{cases} G(X) \triangleq X \times S \\ G(f) \triangleq f \times \mathrm{id}_S \end{cases}$$

$F : \mathbf{Set} \to \mathbf{Set}$	$G : \mathbf{Set} \to \mathbf{Set}$
$\begin{cases} F(X) \triangleq S \times X \\ F(f) \triangleq id_S \times f \end{cases}$	$\begin{cases} G(X) \triangleq X \times S \\ G(f) \triangleq f \times id_S \end{cases}$

For each set $X \in \mathrm{obj}\,\mathbf{Set}$ there is an isomorphism $\theta_X : F(X) \cong G(X)$ given by $\langle \pi_2, \pi_1 \rangle : S \times X \to X \times S$

Those isos don't depend on the particular nature of each X — they are "polymorphic in X".
One way to make this precise is ...

... if we change from X to Y along a function $f : X \to Y$, then we get a commutative square in **Set**

$$
\begin{tikzcd}
F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\theta_Y"'] & G(Y)
\end{tikzcd}
\qquad \text{i.e.} \qquad
\begin{tikzcd}
S \times X \arrow[r, "\langle \pi_2, \pi_1 \rangle"] \arrow[d, "id \times f"] & X \times S \arrow[d, "f \times id"] \\
S \times Y \arrow[r, "\langle \pi_2, \pi_1 \rangle"'] & Y \times S
\end{tikzcd}
$$

we say the family $(\theta_X \mid X \in \mathrm{Obj}\ \mathbf{Set})$ is <u>natural</u> in X

square commutes because:

$$
\begin{aligned}
\langle \pi_2, \pi_1 \rangle ((id \times f)(s, x)) &= \langle \pi_2, \pi_1 \rangle (s, fx) \\
&= (fx, s) \\
&= (f \times id)(x, s)
\end{aligned}
$$

Natural Transformations

<u>Definition</u> Given categories & functors $\mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{D}$

a natural transformation $\boxed{\theta : F \to G}$

is a family of $\mathbb{D}$-morphisms $\theta_X \in \mathbb{D}(FX, GX)$, one for each $\mathbb{C}$-object X, such ~~that~~ for all $\mathbb{C}$-morphisms $f : X \to Y$

$$\begin{tikzcd} FX \arrow[r, "\theta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\theta_Y"'] & GY \end{tikzcd}$$

commutes, i.e.

$$\theta_Y \circ Ff = Gf \circ \theta_X$$

Example

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend right, "F"'] & \mathbf{Mon} \arrow[l, bend right, "U"'] \end{tikzcd}$$

forgetful functor

free monoid functor

There is a natural transformation

$$\eta : \mathrm{Id}_{Set} \to U \circ F$$

where

$$\eta_\Sigma \triangleq \left(\Sigma \xrightarrow{i_\Sigma} List[\Sigma]\right)$$

(for each set Σ)

function mapping each $a \in \Sigma$ to list of length 1 containing a.

Easy to see that

$$\begin{tikzcd} \Sigma \arrow[r, "\eta_\Sigma"] \arrow[d, "f"'] & UF(\Sigma) \arrow[d, "UF(f)"] \\ \Sigma' \arrow[r, "\eta_{\Sigma'}"] & UF(\Sigma') \end{tikzcd}$$

commutes.

Example

Fix a set Σ (of states)

Functor $T \triangleq ((-)\times\Sigma)^{\Sigma} : \mathbf{Set} \to \mathbf{Set}$

Think of elements $c \in T(X) = (X\times\Sigma)^{\Sigma}$ as modelling "computations" that map initial states $s \in \Sigma$ to pairs $c(s) = (x, s')$ where $x \in X$ is the value computed and $s' \in \Sigma$ is the final state

Example

Fix a set Σ (of states)

Functor $T \triangleq ((-)\times\Sigma)^{\Sigma} : \mathbf{Set} \to \mathbf{Set}$

Natural transformation $\mu : T \circ T \to T$

$$\mu_X : T(TX) \longrightarrow TX$$

$\boxed{s \in \Sigma}$ $\boxed{s' \in \Sigma}$

$$\mu_X\, c\, s \triangleq c'(s') \text{ where } c\,s = (c', s')$$

$\boxed{c \in T(TX) = ((X \times \Sigma)^{\Sigma} \times \Sigma)^{\Sigma}}$ $\boxed{c' \in (X \times \Sigma)^{\Sigma}}$

Example

Fix a set Σ (of states)

Functor $T \triangleq ((-)\times\Sigma)^\Sigma : \mathbf{Set} \to \mathbf{Set}$

Natural transformation $\mu : T \circ T \to T$

$$\mu_X : T(TX) \longrightarrow TX$$

$$\mu_X\, c\, s \triangleq c'(s') \text{ where } c\,s = (c', s')$$

Exercise: check that μ_X is natural in X, i.e. if $f : X \to Y$ in $\mathbf{Set}$, then $Tf \circ \mu_X = \mu_Y \circ T(Tf)$

Composing natural transformations

Given functors $F, G, H : \mathbb{C} \to \mathbb{D}$
and natural transformations

$$\theta : F \to G \quad \& \quad \varphi : G \to H$$

we get $\varphi \circ \theta : F \to H$

with $(\varphi \circ \theta)_X = \left(FX \xrightarrow{\theta_X} GX \xrightarrow{\varphi_X} HX \right)$

Check naturality:

$$\begin{aligned} Hf \circ (\varphi \circ \theta)_X &= Hf \circ \varphi_X \circ \theta_X \\ &= \varphi_Y \circ Gf \circ \theta_X = \varphi_Y \circ \theta_Y \circ Ff \\ &= (\varphi \circ \theta)_Y \circ Ff \end{aligned}$$

Identity natural transformation

Given functor $F : \mathbb{C} \to \mathbb{C}$

we get a natural transformation

$$\mathrm{id}_F : F \to F$$

with $(\mathrm{id}_F)_X = \left(FX \xrightarrow{\mathrm{id}_{FX}} FX \right)$

Check naturality:

$$Ff \circ (\mathrm{id}_F)_X = Ff \circ \mathrm{id}_{FX}$$
$$= Ff = \mathrm{id}_{FY} \circ Ff = (\mathrm{id}_F)_Y \circ Ff$$

Easy to see that composition & identities for natural transformations satisfy

$$(\psi \circ \varphi) \circ \theta = \psi \circ (\varphi \circ \theta)$$

$$\mathrm{id}_E \circ \theta = \theta \circ \mathrm{id}_F$$

so we get a category...

Functor categories

Given categories $\mathbb{C}$ & $\mathbb{D}$, the **functor category** $\boxed{\mathbb{D}^{\mathbb{C}}}$ has

- objects are all functors $\mathbb{C} \to \mathbb{D}$
- given $F, G : \mathbb{C} \to \mathbb{D}$, morphisms $F \to G$ in $\mathbb{D}^{\mathbb{C}}$ are natural transformations
- composition & identities as above.

N.B. If $\mathbb{C}$ & $\mathbb{D}$ are small categories, then so is $\mathbb{D}^{\mathbb{C}}$, because

$$\mathrm{obj}(\mathbb{D}^{\mathbb{C}}) \subseteq \sum_{F \in (\mathrm{obj}\,\mathbb{D})^{\mathrm{obj}\,\mathbb{C}}} \prod_{X,Y \in \mathrm{obj}\,\mathbb{C}} \mathbb{D}(FX, FY)$$

$$\mathbb{D}^{\mathbb{C}}(F,G) \subseteq \prod_{X \in \mathrm{obj}\,\mathbb{C}} \mathbb{D}(FX, GX)$$

If $\mathcal{U}$ is a Grothendieck universe then

$$\left.\begin{array}{l} X \in \mathcal{U} \\ F \in \mathcal{U}^X \end{array}\right\} \Rightarrow \left.\begin{array}{l} \boxed{\sum_{x \in X} Fx} \\ \boxed{\prod_{x \in X} Fx} \end{array}\right\} \in \mathcal{U}$$

$$\boxed{\{(x,y) \mid x \in X \ \& \ y \in Fx\}}$$

$$\boxed{\{f \subseteq \textstyle\sum_{x \in X} Fx \mid f \text{ single-valued \& total}\}}$$

Cat is a c.c.c

<u>Theorem</u> There is an application functor

$$\mathrm{app} : \mathbb{D}^{\mathbb{C}} \times \mathbb{C} \longrightarrow \mathbb{D}$$

that gives the exponential of $\mathbb{C}$ & $\mathbb{D}$ in Cat

Definition of $\mathrm{app}: \mathbb{D}^{\mathbb{C}} \times \mathbb{C} \to \mathbb{D}$ on objects:

$$\mathrm{app}(F, X) \triangleq F(X) \qquad \left(\begin{array}{l} F: \mathbb{C} \to \mathbb{D} \\ X \in \mathrm{obj}\,\mathbb{C} \end{array} \right)$$

Definition of $\mathrm{app}: \mathbb{D}^{\mathbb{C}} \times \mathbb{C} \to \mathbb{D}$ on morphisms

$$\mathrm{app}\left((F, X) \xrightarrow{(\theta, f)} (G, Y) \right) \triangleq F(X) \xrightarrow{Ff} F(Y) \xrightarrow{\theta_Y} G(Y)$$

$$= F(X) \xrightarrow{\theta_X} G(X) \xrightarrow{Gf} G(Y)$$

Check: $\begin{cases} \mathrm{app}(id_F, id_X) = id_{F(X)} \\ \mathrm{app}(\varphi \circ \theta, g \circ f) = \mathrm{app}(\varphi, g) \circ \mathrm{app}(\theta, f) \end{cases}$

Definition of currying in $\mathbf{Cat}$:

given functor $F : \mathbb{E} \times \mathbb{C} \to \mathbb{D}$ we get a functor $\mathrm{cur}F : \mathbb{E} \to \mathbb{D}^{\mathbb{C}}$ as follows :

For each $Z \in \mathrm{obj}\,\mathbb{E}$, $\mathrm{cur}F Z : \mathbb{C} \to \mathbb{D}$ is the functor :

$$\mathrm{cur}F Z \left(\begin{array}{c} X \\ \downarrow f \\ X' \end{array} \right) \triangleq \left(\begin{array}{c} F(Z, X) \\ \downarrow F(\mathrm{id}_Z, f) \\ F(Z, X') \end{array} \right)$$

Definition of currying in $\mathbf{Cat}$:

given functor $F : \mathbb{E} \times \mathbb{C} \to \mathbb{D}$

we get a functor $\operatorname{cur} F : \mathbb{E} \to \mathbb{D}^{\mathbb{C}}$

as follows :

For each $Z \xrightarrow{g} Z'$ in $\mathbb{E}$,

$\operatorname{cur} F g : \operatorname{cur} F Z \to \operatorname{cur} F Z'$ is the natural

transformation whose component at $X \in \operatorname{obj} \mathbb{C}$ is

$$\begin{array}{ccc} \operatorname{cur} F Z X & \xrightarrow{\quad (\operatorname{cur} F g)_X \quad} & \operatorname{cur} F Z' X \\ \| & \| & \| \\ F(Z, X) & \xrightarrow{\quad F(g, \mathrm{id}_X) \quad} & F(Z', X) \end{array}$$

check

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Have to check that

$\mathrm{cur}\,F : \mathbb{E} \to \mathbb{D}^{\mathbb{C}}$

is the unique functor $G : \mathbb{E} \to \mathbb{D}^{\mathbb{C}}$

that makes

$$\begin{tikzcd}
\mathbb{E} \times \mathbb{C} \arrow[r, "F"] \arrow[d, "G \times \mathrm{id}_{\mathbb{C}}"'] & \mathbb{D} \\
\mathbb{D}^{\mathbb{C}} \times \mathbb{C} \arrow[ur, "\mathrm{app}"'] &
\end{tikzcd}$$

commute in $\mathbf{Cat}$ (exercise).