Exercise Sheet 4 (graded, 25% of final course mark)

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16:00 MONDAY 14 NOVEMBER

Functors

are the appropriate notion of morphism between categories. Given categories $\mathbb{C}$ & $\mathbb{D}$, a **functor** $\boxed{F : \mathbb{C} \to \mathbb{D}}$ is specified by:

- a function $\mathrm{Obj}\,\mathbb{C} \longrightarrow \mathrm{Obj}\,\mathbb{D}$ whose value at a $\mathbb{C}$-object X is written FX
- for all $\mathbb{C}$-objects X & Y, a function $\mathbb{C}(X,Y) \longrightarrow \mathbb{D}(FX,FY)$ whose value at a $\mathbb{C}$-morphism $f : X \to Y$ is written $Ff : FX \to FY$

satisfying
$$\begin{cases} F(g \circ f) = Fg \circ Ff \\ F(id_X) = id_{FX} \end{cases}$$

Examples of functors

"forgetful" functors from categories of sets-with-structure back to $\mathbf{Set}$

E.g. $\boxed{U : \mathbf{Mon} \to \mathbf{Set}}$

$$\begin{cases} U(M, \circ, e) \triangleq M \\ U\big((M, \circ, e) \xrightarrow{f} (N, \circ, e)\big) \triangleq M \xrightarrow{f} N \end{cases}$$

and similarly $U : \mathbf{Pre} \to \mathbf{Set}$

Examples of functors

Free monoid functor $\boxed{F : \mathbf{Set} \to \mathbf{Mon}}$

Recall free monoid on a set Σ is

$$(\mathrm{List}(\Sigma), @, \mathrm{nil})$$

empty list

finite lists of elements of Σ

list concatenation

Examples of functors

Free monoid functor $\boxed{F : \mathbf{Set} \to \mathbf{Mon}}$

Recall free monoid on a set Σ is

$$F(\Sigma) \triangleq (\mathrm{List}(\Sigma), @, nil)$$

Given $f \in \mathbf{Set}(\Sigma_1, \Sigma_2)$ we get

$F(f) : F(\Sigma_1) \to F(\Sigma_2)$ mapping each list

$l = [a_1, \ldots, a_n] \in \Sigma_1^*$ to $Ffl \triangleq [fa_1, \ldots, fa_n]$

Easy to see that $F(id_\Sigma) = id_{F(\Sigma)}$ &

$$F(g \circ f) = (Fg) \circ (Ff)$$

Examples of functors

If $\mathbb{C}$ is a category with binary products and $X \in \operatorname{obj}\mathbb{C}$, then

$$Y \in \operatorname{obj}\mathbb{C} \mapsto Y \times X$$

extends to a functor $\boxed{(\_) \times X : \mathbb{C} \to \mathbb{C}}$

via $\left(Y \xrightarrow{f} Y'\right) \mapsto \left(Y \times X \xrightarrow{f \times id_X} Y' \times X\right)$

since $\begin{cases} id_Y \times id_X = id_{Y \times X} \\ (g \circ f) \times id_X = (g \times id_X) \circ (f \times id_X) \end{cases}$

[see Ex. Sh. 2, q1c]

Examples of functors

If $\mathbb{C}$ is a cartesian closed category and $X \in \mathrm{obj}\,\mathbb{C}$, then

$$Y \in \mathrm{obj}\,\mathbb{C} \mapsto Y^X$$

extends to a functor $\boxed{(\_)^X : \mathbb{C} \to \mathbb{C}}$

via $(Y \xrightarrow{f} Y') \mapsto (Y^X \xrightarrow{f^X} Y'^X)$

$$f^X = \mathrm{cur}(f \circ \mathrm{app})$$

Since $\begin{cases} \mathrm{id}^X = \mathrm{id} \\ (g \circ f)^X = g^X \circ f^X \end{cases}$ [see Ex. Sh. 3, q 4]

Contravariance

A functor $F : \mathbb{C}^{op} \to \mathbb{D}$ is called a **contravariant functor from $\mathbb{C}$ to $\mathbb{D}$**

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbb{C}$

then $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathbb{C}^{op}$

so

$$\begin{tikzcd} FX & FY \arrow[l, "Ff"'] & FZ \arrow[l, "Fg"'] \arrow[ll, bend left] \end{tikzcd} \quad \text{in } \mathbb{D}$$

$$F(g \circ f) = Ff \circ Fg$$

Example of contravariant functor

If $\mathbb{C}$ is a cartesian closed category and $X \in \mathrm{obj}\,\mathbb{C}$, then

$$Y \in \mathrm{obj}\,\mathbb{C} \mapsto X^Y$$

extends to a functor $\boxed{X^{(-)} : \mathbb{C}^{op} \to \mathbb{C}}$

via $(Y \xrightarrow{f} Y') \mapsto \left(X^Y \xleftarrow{X^f} X^{Y'} \right)$

$$X^f = cur(app \circ (id \times f))$$

Since $\begin{cases} X^{id} = id \\ X^{g \circ f} = X^f \circ X^g \end{cases}$ [see Ex. Sh. 3, q5]

Note ~~that~~ since a functor $F : \mathbb{C} \to \mathbb{D}$ preserves domains, codomains, composition

it sends commutative diagrams in $\mathbb{C}$ to commutative diagrams in $\mathbb{D}$

E.g.

$$
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[d, bend right, "h"'] & Y \arrow[dl, bend left, "g"] \\
Z &
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
FX \arrow[r, bend left, "Ff"] \arrow[d, bend right, "Fh"'] & FY \arrow[dl, bend left, "Fg"] \\
FZ &
\end{tikzcd}
$$

$$Fh = F(g \circ f) = Fg \circ Ff$$

Note ~~that~~ since a functor $F : \mathbb{C} \to \mathbb{D}$ preserves domains, codomains, composition & identities,

it sends isomorphisms in $\mathbb{C}$ to isos in $\mathbb{D}$ because

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\mathrm{id}_X"'] & Y \arrow[d, "g"] \arrow[dr, "\mathrm{id}_Y"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[dr, "\mathrm{id}_{FX}"'] & FY \arrow[d, "Fg"] \arrow[dr, "\mathrm{id}_{FY}"] & \\
& FX \arrow[r, "Ff"'] & FY
\end{tikzcd}
$$

So $\boxed{F(f^{-1}) = (Ff)^{-1}}$

Composing functors

Given functors

$$\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$$

we get a functor $\boxed{G \circ F : \mathbb{C} \to \mathbb{E}}$

$$G \circ F \begin{pmatrix} X \\ \downarrow f \\ Y \end{pmatrix} \triangleq \begin{array}{c} G(FX) \\ \downarrow G(Ff) \\ G(FY) \end{array}$$

(this preserves composition & identities because F & G do so)

Identity functor

on a category $\mathbb{C}$ is

$$\boxed{\mathrm{Id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}}$$

where

$$\mathrm{Id}_{\mathbb{C}}\left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array}\right) \triangleq \begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

Functor composition satisfies the usual category laws

associativity $H \circ (G \circ F) = (H \circ G) \circ F$

unity $Id_{\mathbb{D}} \circ F = F = F \circ Id_{\mathbb{C}}$

So we can get categories whose
objects are categories
morphisms are functors
but we have to be a bit carful about size...

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Russell's Paradox

Urestricted use of set comprehension

$\{x \mid \varphi(x)\}$ the set of all objects x that have property $\varphi(x)$

leads to contradiction (a proof of false),

since $R \triangleq \{x \mid x \notin x\}$

satisfies $R \in R \iff R \notin R$,

which is logically equivalent to false.

Size

We can't form the "set of <u>all</u> sets"
or "the category of <u>all</u> categories"

because we assume

set membership is a well-founded relation — there can be no infinite sequence of sets $X_0, X_1, X_2, \ldots$ with

$$\cdots X_{n+1} \in X_n \in \cdots \in X_2 \in X_1 \in X_0$$

so in particular, there is no set X with $X \in X$

Size

We can't form the "set of <u>all</u> sets" or "the category of <u>all</u> categories"

but we do assume there are (lots of) big sets

$$\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$$

where each $\mathcal{U}_i$ is a Grothendieck universe...

A Grothendieck Universe, $\mathcal{U}$

is a set of sets satisfying

- $x \in y \in \mathcal{U} \Rightarrow x \in \mathcal{U}$
- $x, y \in \mathcal{U} \Rightarrow \{x, y\} \in \mathcal{U}$
- $x \in \mathcal{U} \Rightarrow Px \triangleq \{y \mid y \subseteq x\} \in \mathcal{U}$
- $X \in \mathcal{U}$ & $F \in \mathcal{U}^X \Rightarrow \{y \mid (\exists x \in X)\, y \in Fx\} \in \mathcal{U}$

and hence also

$$x, y \in \mathcal{U} \Rightarrow x \times y \in \mathcal{U} \;\&\; y^x \in \mathcal{U}.$$

The above properties are satisfied by $\mathcal{U} = \emptyset$, but we will always assume

- (axiom of infinity) $\mathbb{N} \in \mathcal{U}$

Size

We assume there is an infinite sequence

$$\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$$

of bigger & bigger Grothendieck universes.

and revise our previous definition of "the" category of sets:

$\mathbf{Set}_i \triangleq$ category whose objects are all the elements of $\mathcal{U}_i$ and with $\mathbf{Set}_i(X, Y) = Y^X$ = all functions from X to Y

$\mathbf{Set} \triangleq \mathbf{Set}_0$ – its objects are called **small sets** (and other sets we call **large**)

Size

Set is the category of small sets.

Definition A category $\mathbb{C}$ is locally small if for all $X, Y \in \text{obj}\,\mathbb{C}$, $\mathbb{C}(X,Y) \in \mathbf{Set}$.

$\mathbb{C}$ is a small category if it is both locally small & $\text{obj}\,\mathbb{C} \in \mathbf{Set}$

E.g. **Set**, **Pre**, **Mon** are all locally small (but not small).

Each $P \in \mathbf{Pre}$ & $M \in \mathbf{Mon}$ determines a small category.

The category of small categories, $\mathbf{Cat}$

- objects are all small categories
- morphisms $\mathbf{Cat}(\mathbb{C}, \mathbb{D})$ are all functors $F : \mathbb{C} \to \mathbb{D}$
- Composition & identities – for functors, as before