

Exercise Sheet 4 (graded,
25% of final course mark)

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16:00 MONDAY 14 NOVEMBER

Functors

are the appropriate notion of morphism between categories. Given categories \mathbb{C} & \mathbb{D} , a **functor** $F : \mathbb{C} \rightarrow \mathbb{D}$ is specified by:

- a function $\text{Obj } \mathbb{C} \rightarrow \text{Obj } \mathbb{D}$ whose value at a \mathbb{C} -object X is written FX
- for all \mathbb{C} -objects X & Y , a function $\mathbb{C}(X, Y) \rightarrow \mathbb{D}(FX, FY)$ whose value at a \mathbb{C} -morphism $f : X \rightarrow Y$ is written $Ff : FX \rightarrow FY$

satisfying $\left\{ \begin{array}{l} F(g \circ f) = Fg \circ Ff \\ F(\text{id}_X) = \text{id}_{FX} \end{array} \right.$

Examples of functors

"forgetful" functors from categories of sets-with-structure back to **Set**

E.g. $U : \text{Mon} \rightarrow \text{Set}$

$$\begin{cases} U(M, \cdot, e) \triangleq M \\ U((M, \cdot, e) \xrightarrow{f} (N, \cdot, e)) \triangleq M \xrightarrow{f} N \end{cases}$$

and similarly $U : \text{Pre} \rightarrow \text{Set}$

Examples of functors

Free monoid functor

$$F : \text{Set} \rightarrow \text{Mon}$$

Recall free monoid on a set Σ is

$$(\text{List}(\Sigma), @, \text{nil})$$

empty list

finite lists of
elements of Σ

list concatenation

Examples of functors

Free monoid functor

$$F : \text{Set} \rightarrow \text{Mon}$$

Recall free monoid on a set Σ is

$$F(\Sigma) \triangleq (\text{List}(\Sigma), @, \text{nil})$$

Given $f \in \text{Set}(\Sigma_1, \Sigma_2)$ we get

$$F(f) : F(\Sigma_1) \rightarrow F(\Sigma_2) \quad \text{mapping each list}$$
$$l = [a_1, \dots, a_n] \in \Sigma_1^* \quad \text{to} \quad Ff l \triangleq [fa_1, \dots, fa_n]$$

Easy to see that $F(\text{id}_\Sigma) = \text{id}_{F(\Sigma)}$ &

$$F(g \circ f) = (Fg) \circ (Ff)$$

Examples of functors

If \mathcal{C} is a category with binary products and $X \in \text{Obj } \mathcal{C}$, then

$$Y \in \text{Obj } \mathcal{C} \mapsto Y \times X$$

extends to a functor

$$(-) \times X : \mathcal{C} \rightarrow \mathcal{C}$$

$$\text{via } (Y \xrightarrow{f} Y') \mapsto (Y \times X \xrightarrow{f \times \text{id}_X} Y' \times X)$$

$$\text{since } \begin{cases} \text{id}_Y \times \text{id}_X = \text{id}_{Y \times X} \\ (g \circ f) \times \text{id}_X = (g \times \text{id}_X) \circ (f \times \text{id}_X) \end{cases}$$

[see Ex. Sh. 2, q. 1c]

Examples of functors

If \mathcal{C} is a cartesian closed category and $X \in \text{Obj } \mathcal{C}$, then

$$Y \in \text{Obj } \mathcal{C} \mapsto Y^X$$

extends to a functor

$$(-)^X : \mathcal{C} \rightarrow \mathcal{C}$$

$$\text{via } (Y \xrightarrow{f} Y') \mapsto (Y^X \xrightarrow{f^X} Y'^X)$$

$$\text{Since } \begin{cases} \text{id}^X = \text{id} \\ (g \circ f)^X = g^X \circ f^X \end{cases} \quad \begin{array}{l} \parallel \\ \text{cur}(f \circ \text{app}) \end{array}$$

[see Ex. Sh. 3, p. 4]

Contravariance

A functor $F: \mathbb{C}^{\text{op}} \rightarrow \mathbb{D}$ is called a **contravariant functor** from \mathbb{C} to \mathbb{D}

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathbb{C}

then $X \xleftarrow{f} Y \xleftarrow{g} Z$ in \mathbb{C}^{op}

so

$F X \xleftarrow{F f} F Y \xleftarrow{F g} F Z$ in \mathbb{D}



$$F(g \circ f) = Ff \circ Fg$$

Example of contravariant functor

If \mathcal{C} is a cartesian closed category and $X \in \text{Obj } \mathcal{C}$, then

$$Y \in \text{Obj } \mathcal{C} \mapsto X^Y$$

extends to a functor

$$X^{(-)} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$$

via $(Y \xrightarrow{f} Y') \mapsto (X^Y \xleftarrow{X^f} X^{Y'})$

$$\text{Cur}(\text{app} \circ (\text{id} \times f))$$

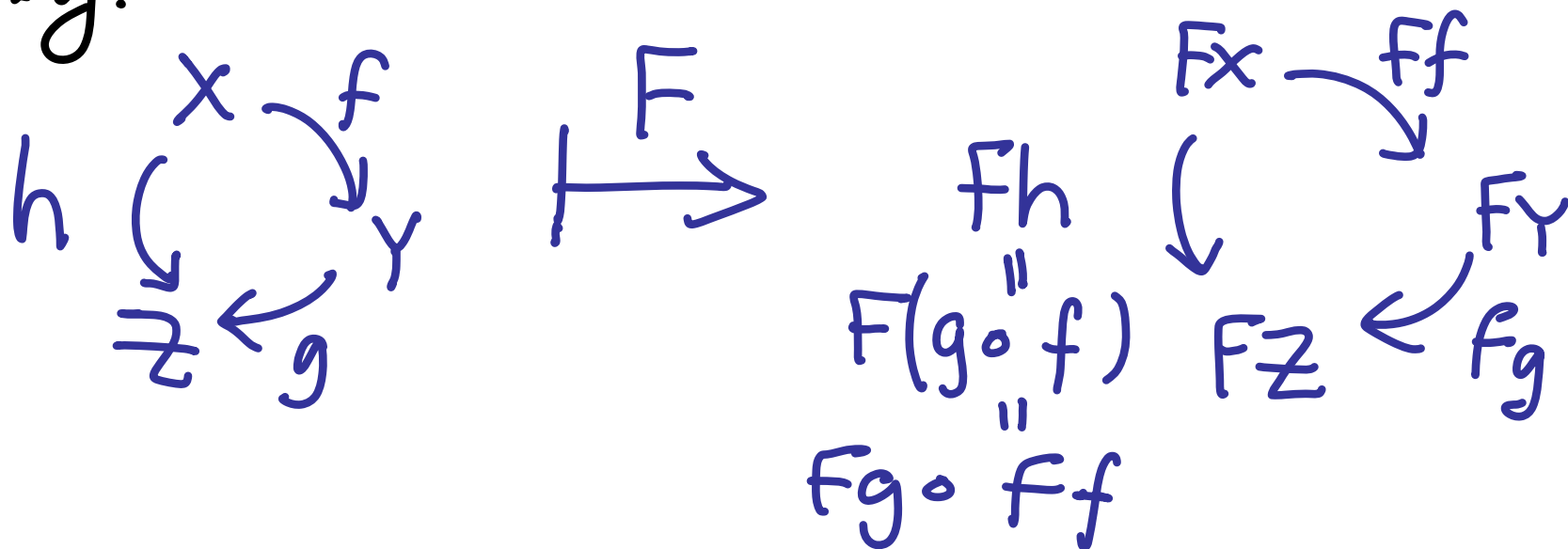
Since $\begin{cases} X^{\text{id}} = \text{id} \\ X^{g \circ f} = X^f \circ X^g \end{cases}$

[see Ex. Sh. 3, 95]

Note that since a functor $F: \mathbb{C} \rightarrow \mathbb{D}$ preserves domains, codomains, composition

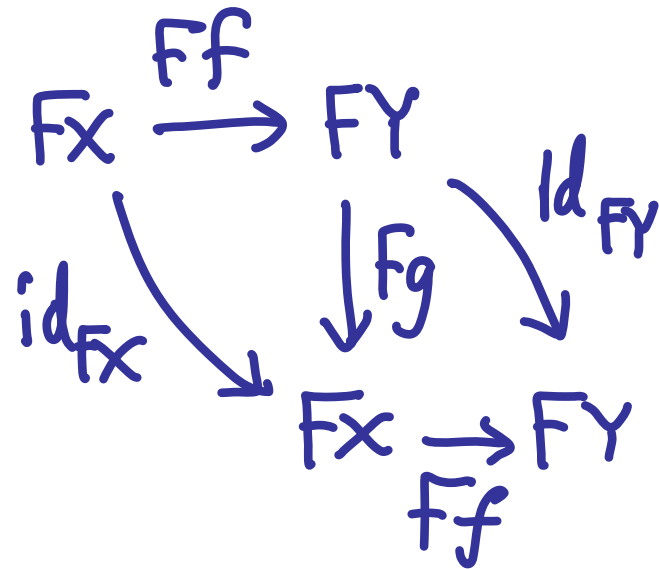
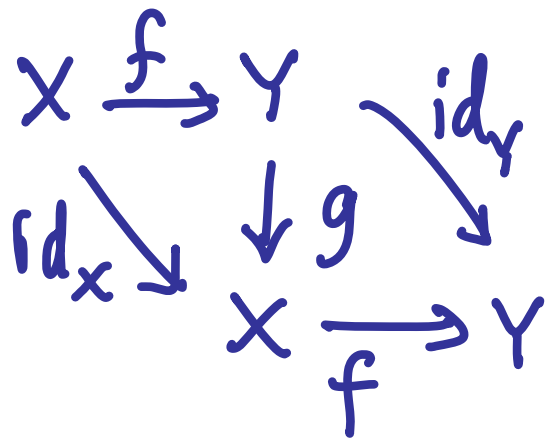
it sends commutative diagrams in \mathbb{C} to commutative diagrams in \mathbb{D}

E.g.



Note that since a functor $F: \mathbb{C} \rightarrow \mathbb{D}$ preserves domains, codomains, composition & identities,

it sends isomorphisms in \mathbb{C} to isos in \mathbb{D} because



So $F(f^{-1}) = (Ff)^{-1}$

Composing functors

Given functors

$$\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$$

we get a functor

$$G \circ F : \mathbb{C} \rightarrow \mathbb{E}$$

$$G \circ F \left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \right) \triangleq \begin{array}{c} G(FX) \\ \downarrow G(f) \\ G(FY) \end{array}$$

(this preserves composition & identities because F & G do so)

Identity functor

on a category \mathcal{C} is

$$\text{Id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$$

where

$$\text{Id}_{\mathcal{C}} \left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \right) \triangleq$$

$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

Functor composition satisfies the usual
Category laws

associativity

$$H \circ (G \circ F) = (H \circ G) \circ F$$

unity

$$\text{Id}_D \circ F = F = F \circ \text{Id}_C$$

So we can get categories whose
objects are categories
morphisms are functors

but we have to be a bit careful about **Size...**

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but we have to be a bit careful about size...

Russell's Paradox

Unrestricted use of **set comprehension**

$\{ x \mid \varphi(x) \}$ the set of all objects x that have property $\varphi(x)$

leads to contradiction (a proof of false),

Since $R \triangleq \{ x \mid x \notin x \}$

satisfies $R \in R \iff R \notin R$,

which is logically equivalent to false.

Size

We can't form the "set of all sets"
or "the category of all categories"
because we assume

Set membership is a well-founded
relation — there can be no infinite
sequence of sets X_0, X_1, X_2, \dots with
 $\dots X_{n+1} \in X_n \in \dots \in X_2 \in X_1 \in X_0$

so in particular, there is no set X with $X \in X$

Size

We can't form the "set of all sets"
or "the category of all categories"

but we do assume there are (lots of)
big sets

$$U_0 \in U_1 \in U_2 \in \dots$$

where each U_i is a Grothendieck
universe ...

A Grothendieck Universe, \mathcal{U}

is a set of sets satisfying

- $x \in y \in \mathcal{U} \Rightarrow x \in \mathcal{U}$
 - $x, y \in \mathcal{U} \Rightarrow \{x, y\} \in \mathcal{U}$
 - $x \in \mathcal{U} \Rightarrow \mathcal{P}x \triangleq \{y \mid y \subseteq x\} \in \mathcal{U}$
 - $x \in \mathcal{U} \ \& \ F \in \mathcal{U}^x \Rightarrow \{y \mid (\exists x \in x) y \in Fx\} \in \mathcal{U}$
- and hence also

$$x, y \in \mathcal{U} \Rightarrow x \times y \in \mathcal{U} \ \& \ y^x \in \mathcal{U}.$$

The above properties are satisfied by $\mathcal{U} = \emptyset$, but we will always assume

- (axiom of infinity) $\mathbb{N} \in \mathcal{U}$

Size

We assume there is an infinite sequence

$$U_0 \in U_1 \in U_2 \in \dots$$

of bigger & bigger Grothendieck universes.

and revise our previous definition of "the" category of sets:

$\text{Set}_i \triangleq$ category whose objects are all the elements of U_i and with $\text{Set}_i(x, y) = Y^X =$ all functions from X to Y

$\text{Set} \triangleq \text{Set}_0$ - its objects are called **small sets** (and other sets we call **large**)

Size

Set is the category of small sets.

Definition A category \mathcal{C} is locally small if for all $X, Y \in \text{obj } \mathcal{C}$, $\mathcal{C}(X, Y) \in \text{Set}$

\mathcal{C} is a small category if it is both locally small & $\text{obj } \mathcal{C} \in \text{Set}$

E.g. Set, Pre, Mon are all locally small (but not small).

Each $P \in \text{Pre}$ & $M \in \text{Mon}$ determines a small category.

The category of small categories, \mathbf{Cat}

- objects are all small categories
- morphisms $\mathbf{Cat}(\mathbb{C}, \mathbb{D})$ are all functors $F: \mathbb{C} \rightarrow \mathbb{D}$
- Composition & identities - for functors, as before