MPhil ACS/CST Part III 2016

Module L108

CATEGORY THEORY & LOGIC

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What is category theory?

What we are probably seeking is a "purer" view of functions: a theory of functions in themselves, not a theory of functions derived from sets. What, then, is a pure theory of functions? Answer: category theory

Dana Scott, Relating theories of the λ-calculus, p 406

What is category theory?

SET THEORY gives an element-oriented account of mathematical structure

whereas CATEGORY THEORY takes a function-oriented view: understand structures not via their elements, but by how they transform, i.e. via "morphisms"

(Both are part of LOGIC, broadly construed.)

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MacLANE

CONTENTS

Introduction. The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its "dual"

Presented to the Society, September 8, 1942; received by the editors May 15, 1945.

1.4

Category Theory emerges

1945 Eilenberg† & MacLane†, "General Theory of Natural Equivalences", Trans AMS 58, 231–294.
Algebraic topology, abstract algebra

50s Grothendieck† algebraic geometry

60s Lawvere logic & foundations

70s Joyal & Tierney topos theory

80s Dana Scott Lambek†
Semantics linguistics

Category Theory & Computer Science

"Category Theory has ... become part of the standard "tool-box" in many areas of theoretical informatics, from programming languages to automata, from process calculi to Type Theory."

Dagstuhl Perspectives Workshop on Categorical Methods at the Crossroads, April 2014

This course

CT basic concepts:
- adjunction
- natural transformation
- functor
- category

applied to:
- typed λ-calculus
- equational logic
- first order logic

Definition

A category $\mathbb{C}$ is specified by

- a collection $\mathrm{Obj}\,\mathbb{C}$ of $\mathbb{C}$-objects $X, Y, Z, \ldots$
- for each $X, Y \in \mathrm{Obj}\,\mathbb{C}$, a collection $\mathbb{C}(X,Y)$ of $\mathbb{C}$-morphisms from X to Y
- an operation assigning to each $X \in \mathrm{Obj}\,\mathbb{C}$, an identity morphism $id_X \in \mathbb{C}(X,X)$
- an operation assigning to each $f \in \mathbb{C}(X,Y)$ & $g \in \mathbb{C}(Y,Z)$ a composition $g \circ f \in \mathbb{C}(X,Z)$

satisfying ...

Definition, cont.

Satisfying ...

Associativity: for all $f \in \mathbb{C}(X,Y)$, $g \in \mathbb{C}(Y,Z)$ & $h \in \mathbb{C}(Z,W)$

$$\boxed{h \circ (g \circ f) = (h \circ g) \circ f}$$

Unity: for all $f \in \mathbb{C}(X,Y)$

$$\boxed{id_Y \circ f = f = f \circ id_X}$$

Example: category of sets, **Set**

- $\mathrm{Obj}\,\mathbf{Set}$ = some fixed universe of sets
- $\mathbf{Set}(X, Y) =$
  $\{f \subseteq X \times Y \mid f$ is single-valued & total$\}$

Cartesian product consists of all ordered pairs (x, y) with $x \in X$ & $y \in Y$

$(x, y) = (x', y') \iff x = x' \wedge y = y'$

Example: category of sets, **Set**

- Obj **Set** = some fixed universe of sets
- $\mathbf{Set}(X, Y) =$
  $\{f \subseteq X \times Y \mid f$ is single-valued & total$\}$

single-valued:

$$\boxed{(\forall x \in X)(\forall y, y' \in Y)\ (x, y) \in f \wedge (x, y') \in f \Rightarrow y = y'}$$

total:

$$\boxed{(\forall x \in X)(\exists y \in Y)\ (x, y) \in f}$$

Example: category of sets, **Set**

- $\mathrm{Obj}\,\mathbf{Set}$ = some fixed universe of sets
- $\mathbf{Set}(X,Y) =$
  $\{f \subseteq X \times Y \mid f$ is single-valued & total$\}$
- $id_X \triangleq \{(x,x) \mid x \in X\}$
- Composition of $f \in \mathbf{Set}(X,Y)$ & $g \in \mathbf{Set}(Y,Z)$ is
  $$g \circ f \triangleq \{(x,z) \mid (\exists y \in Y)\,(x,y) \in f \wedge (y,z) \in g\}$$

[Check associativity & unity properties hold.]

Example: category of sets, **Set**

Notation:

given $f \in \mathbf{Set}(X, Y)$ & $x \in X$

it's usual to write $f x$ (or $f(x)$)

for the unique $y \in Y$ with $(x, y) \in f$.

Thus

$$\mathrm{id}_X\, x = x$$

$$(g \circ f)\, x = g(f x)$$

Definition

A **category** $\mathbb{C}$ is specified by

- a collection $\mathrm{Obj}\,\mathbb{C}$ of **$\mathbb{C}$-objects** $X, Y, Z, \ldots$
- for each $X, Y \in \mathrm{Obj}\,\mathbb{C}$, a collection $\mathbb{C}(X, Y)$ of **$\mathbb{C}$-morphisms from** X **to** Y
- an operation assigning to each $X \in \mathrm{Obj}\,\mathbb{C}$, an **identity morphism** $\mathrm{id}_X \in \mathbb{C}(X, X)$
- an operation assigning to each $f \in \mathbb{C}(X, Y)$ & $g \in \mathbb{C}(Y, Z)$ a **composition** $g \circ f \in \mathbb{C}(X, Z)$

satisfying ...

Associated notation & terminology

$\boxed{f : X \to Y}$ or $\boxed{X \xrightarrow{f} Y}$ means $f \in \mathbb{C}(X, Y)$

in which case we say

X is the domain of f

Y is the codomain of f

and write

$X = \mathrm{dom}\, f$

$Y = \mathrm{cod}\, f$

(which category $\mathbb{C}$ we are referring to is left implicit)

Commutative diagrams

in a category $\mathbb{C}$ are

diagram: directed graphs whose vertices are $\mathbb{C}$-objects and whose edges are $\mathbb{C}$-morphisms

commutative: such that any two finite paths between two vertices determine equal morphisms under composition

Examples of commutative diagrams

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, bend right, "f"'] & Y \arrow[d, "\mathrm{id}_Y"] \arrow[dr, bend left, "g"] & \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[rr, "h\circ(g\circ f)"] \arrow[d, "f"'] & & W \\
Y \arrow[urr, "h\circ g"] \arrow[rr, "g"'] & & Z \arrow[u, "h"']
\end{tikzcd}
$$

Alternative notation

I'll often write

$\mathbb{C}$ for $\operatorname{ob} \mathbb{C}$

id for id_x

Some people write

1_x for id_x

gf for $g \circ f$

$f ; g$, or fg for $g \circ f$

Alternative definition of category

(The definition I gave is "dependent-type friendly".)

See [Awodey, Def^n 1.1] for an alternative (equivalent) formulation.

(One gives the whole collection of morphisms $\mathrm{Mor}\,\mathbb{C}$ (equivalent to $\sum_{X,Y \in \mathrm{Obj}\,\mathbb{C}} \mathbb{C}(X,Y)$ in our definition) plus operations $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor}\,\mathbb{C} \to \mathrm{Obj}\,\mathbb{C}$. Composition is a partial op^n $\mathrm{Mor}\,\mathbb{C} \times \mathrm{Mor}\,\mathbb{C} \to \mathrm{Mor}\,\mathbb{C}$ defined at (f,g) iff $\mathrm{cod}\, f = \mathrm{dom}\, g$.)