

MPhil ACS/CST Part III 2016

Module L108

# CATEGORY THEORY & LOGIC

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# What is category theory?

What we are probably seeking is a "purer" view of **functions**: a theory of functions in themselves, not a theory of functions derived from sets. What, then, is a pure theory of functions? Answer: **category theory**

Dana Scott, Relating theories of the  $\lambda$ -calculus, p 406

# What is category theory?

SET THEORY gives an **element-oriented** account of mathematical structure

whereas CATEGORY THEORY takes a **function-oriented** view : understand structures not via their elements, but by how they transform, i.e. via "**morphisms**".

(Both are part of LOGIC, broadly construed.)

# GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

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**Introduction.** The subject matter of this paper is best explained by an example, such as that of the relation between a vector space  $L$  and its “dual”

Presented to the Society, September 8, 1942; received by the editors May 15, 1945.

# Category Theory emerges

1945 Eilenberg<sup>+</sup> & MacLane<sup>+</sup>, "General Theory of Natural Equivalences", Trans AMS 58, 231-294.

Algebraic topology, abstract algebra

50s Grothendieck<sup>+</sup> algebraic geometry

60s Lawvere logic & foundations

70s Joyal & Tierney topos theory

80s Dana Scott Lambek<sup>+</sup>  
Semantics linguistics

# Category Theory & Computer Science

"Category Theory has... become part of the standard "tool-box" in many areas of theoretical informatics, from programming languages to automata, from process calculi to Type Theory."

Dagstuhl Perspectives Workshop on Categorical Methods at the Crossroads, April 2014

# This course

CT  
basic  
concepts

adjunction  
natural transformation  
functor  
category

applied to

typed  $\lambda$ -calculus  
equational logic  
first order logic

# Definition

A **Category**  $\mathcal{C}$  is specified by

- a collection  $\text{Obj } \mathcal{C}$  of  **$\mathcal{C}$ -objects**  $X, Y, Z, \dots$
- for each  $X, Y \in \text{Obj } \mathcal{C}$ , a collection  $\mathcal{C}(X, Y)$  of  **$\mathcal{C}$ -morphisms from  $X$  to  $Y$**
- an operation assigning to each  $X \in \text{Obj } \mathcal{C}$ , an **identity morphism**  $\text{id}_X \in \mathcal{C}(X, X)$
- an operation assigning to each  $f \in \mathcal{C}(X, Y)$  &  $g \in \mathcal{C}(Y, Z)$  a **composition**  $g \circ f \in \mathcal{C}(X, Z)$

satisfying ...



# Definition, cont.

Satisfying ...

**Associativity:** for all  $f \in \mathcal{C}(X, Y)$ ,  $g \in \mathcal{C}(Y, Z)$   
&  $h \in \mathcal{C}(Z, W)$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

**Unity:** for all  $f \in \mathcal{C}(X, Y)$

$$\text{id}_Y \circ f = f = f \circ \text{id}_X$$

# Example: category of sets, Set

- Obj Set = some fixed universe of sets
- $\text{Set}(X, Y) =$   
 $\{f \subseteq X \times Y \mid f \text{ is single-valued \& total}\}$

Cartesian product consists of all ordered pairs  $(x, y)$  with  $x \in X$  &  $y \in Y$   
 $(x, y) = (x', y') \iff x = x' \wedge y = y'$

# Example: category of sets, Set

- Obj Set = some fixed universe of sets
- $\text{Set}(X, Y) = \{f \subseteq X \times Y \mid f \text{ is single-valued \& total}\}$

single-valued:

$$(\forall x \in X)(\forall y, y' \in Y) (x, y) \in f \wedge (x, y') \in f \Rightarrow y = y'$$

total:

$$(\forall x \in X)(\exists y \in Y) (x, y) \in f$$

# Example: category of sets, Set

- $\text{Obj Set} =$  some fixed universe of sets
- $\text{Set}(X, Y) =$   
 $\{f \subseteq X \times Y \mid f \text{ is single-valued \& total}\}$
- $\text{id}_X \triangleq \{(x, x) \mid x \in X\}$
- Composition of  $f \in \text{Set}(X, Y)$  &  $g \in \text{Set}(Y, Z)$  is  
 $g \circ f \triangleq \{(x, z) \mid (\exists y \in Y) (x, y) \in f \wedge (y, z) \in g\}$

[Check associativity & unity properties hold.]

# Example: category of sets, Set

## Notation:

given  $f \in \text{Set}(X, Y)$  &  $x \in X$

it's usual to write  $fx$  (or  $f(x)$ )

for the unique  $y \in Y$  with  $(x, y) \in f$ .

Thus  $\text{id}_x x = x$

$$(g \circ f) x = g(fx)$$

# Definition

A **Category**  $\mathcal{C}$  is specified by

- a collection  $\text{Obj } \mathcal{C}$  of  **$\mathcal{C}$ -objects**  $X, Y, Z, \dots$
- for each  $X, Y \in \text{Obj } \mathcal{C}$ , a collection  $\mathcal{C}(X, Y)$  of  **$\mathcal{C}$ -morphisms from  $X$  to  $Y$**
- an operation assigning to each  $X \in \text{Obj } \mathcal{C}$ , an **identity morphism**  $\text{id}_X \in \mathcal{C}(X, X)$
- an operation assigning to each  $f \in \mathcal{C}(X, Y)$  &  $g \in \mathcal{C}(Y, Z)$  a **composition**  $g \circ f \in \mathcal{C}(X, Z)$

satisfying ...

# Associated notation & terminology

$f: X \rightarrow Y$  or  $X \xrightarrow{f} Y$  means  $f \in \mathcal{C}(X, Y)$

↳ in which case we say

$X$  is the **domain** of  $f$

$Y$  is the **codomain** of  $f$

and write

$$X = \text{dom } f$$

$$Y = \text{cod } f$$

(which category  $\mathcal{C}$  we are referring to is left implicit)

# Commutative diagrams

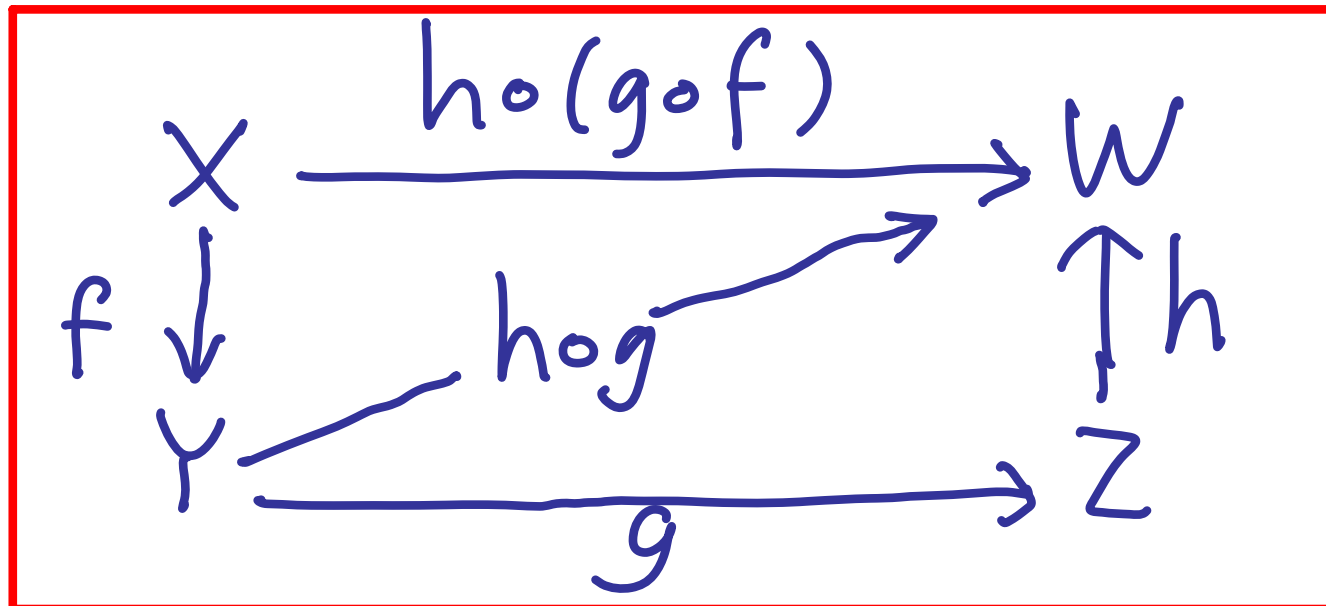
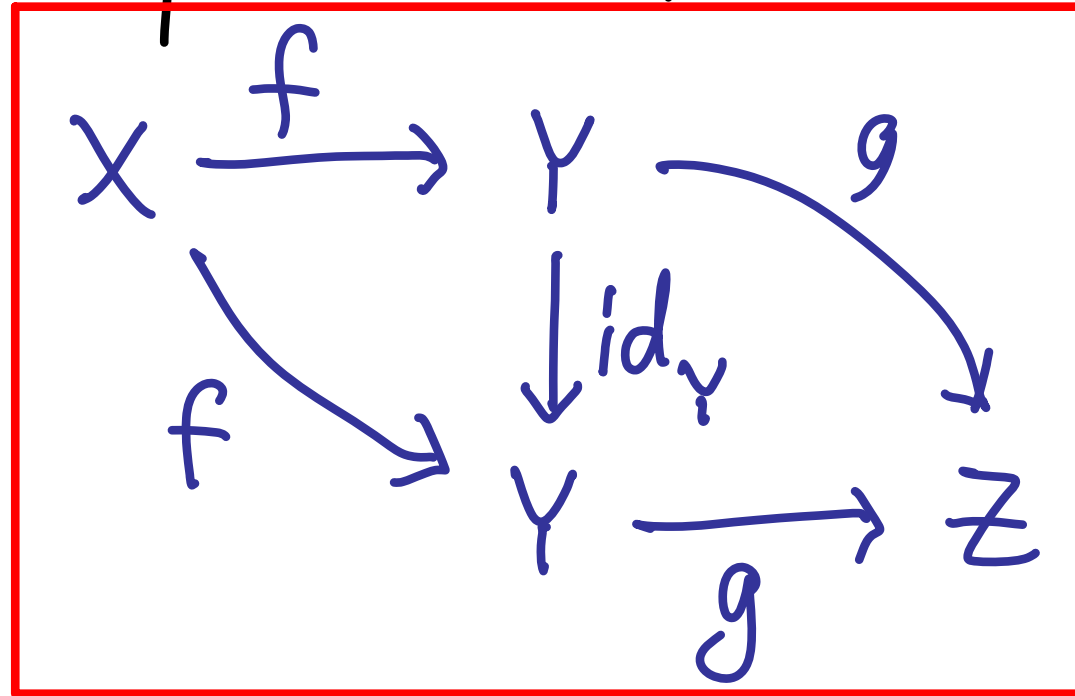
in a category  $\mathcal{C}$  are

diagram { directed graphs whose vertices are  $\mathcal{C}$ -objects and whose edges are  $\mathcal{C}$ -morphisms

Com-  
mutative { such that any two finite paths between two vertices determine equal morphisms under composition



# Examples of commutative diagrams



# Alternative notation

I'll often write

$\mathbb{C}$  for  $\text{Obj } \mathbb{C}$

$\text{id}$  for  $\text{id}_x$

Some people write

$1_x$  for  $\text{id}_x$

$gf$  for  $g \circ f$

$f; g$ , or  $fg$  for  $g \circ f$

# Alternative definition of category

(The definition I gave is "dependent-type friendly".)

See [Awodey, Def<sup>n</sup> 1.1] for an alternative (equivalent) formulation.

(One gives the whole collection of morphisms  $\text{Mor } \mathcal{C}$  (equivalent to  $\sum_{X, Y \in \text{Obj } \mathcal{C}} \mathcal{C}(X, Y)$  in our definition) plus operations  $\text{dom}, \text{cod} : \text{Mor } \mathcal{C} \rightarrow \text{Obj } \mathcal{C}$ . Composition is a partial op<sup>n</sup>  $\text{Mor } \mathcal{C} \times \text{Mor } \mathcal{C} \rightarrow \text{Mor } \mathcal{C}$  defined at  $(f, g)$  iff  $\text{cod } f = \text{dom } g$ .)