Question 1
(a) There is at most one natural transformation $F \to G$; and there is one iff 
$(\forall x \in P) F x \leq_Q G x$.
(b) Given monotone functions $F, G : (P, \leq_P) \to (Q, \leq_Q)$, $G$ is right adjoint to $F$ iff 
$(\forall x \in P)(\forall y \in Q) F x \leq_Q y \iff x \leq_P G y$.

Question 2
(a) If $B \subseteq B'$, then for all $x \in f^{-1}B$, $f x \in B$, so $f x \in B'$, so $x \in f^{-1}B'$; therefore 
$f^{-1}B \subseteq f^{-1}B'$.

(b) It is immediate from the definitions of $\exists_f(A)$ and $\forall_f(A)$ that 
$A \subseteq A' \implies \exists_f A \subseteq \exists_f A' \land \forall_f A \subseteq \forall_f A'$
so that $\exists_f$ and $\forall_f$ are monotone functions. So to see that $\exists_f$ and $\forall_f$ are respectively 
left and right adjoints to $f^{-1}$, by the answer to question 1b we have to prove for all 
$A \in \text{Pow } X$ and $B \in \text{Pow } Y$
\[ \exists_f A \subseteq B \iff A \subseteq f^{-1}B \tag{4} \]
\[ f^{-1}B \subseteq A \iff B \subseteq \forall_f A \tag{5} \]
For (4): $\exists_f A \subseteq B \iff (\forall y \in Y)((\exists x \in X) f x = y \land x \in A) \implies y \in B$ .
\[ \iff (\forall y \in Y, x \in X) f x = y \land x \in A \implies y \in B \]
\[ \iff (\forall x \in X) x \in A \implies (\forall y \in Y) f x = y \implies y \in B \]
\[ \iff (\forall x \in X) x \in A \implies f x \in B \]
\[ \iff (\forall x \in X) x \in A \implies x \in f^{-1}B \]
\[ \iff A \subseteq f^{-1}B \]
For (5): $B \subseteq \forall_f A \iff (\forall y \in Y) y \in B \implies (\forall x \in X) f x = y \implies x \in A$.
\[ \iff (\forall y \in Y, x \in X) y \in B \land f x = y \implies x \in A \]
\[ \iff (\forall x \in X) f x \in B \implies x \in A \]
\[ \iff (\forall x \in X) x \in f^{-1}B \implies x \in A \]
\[ \iff f^{-1}B \subseteq A \]

Question 3
(a) The universal property of (1) in $C$ is exactly the same as the universal property for a 
product of $(Y, f)$ and $(Z, g)$ in the slice category $C/X$. 
(b) If 1 is terminal in $\mathbf{C}$, then a pullback for $X \xrightarrow{\pi_1} 1 \xleftarrow{\pi_2} Y$ has the same universal property as a product for $X$ and $Y$ in $\mathbf{C}$.

(c) Given $h : (Z, g) \to (Z', g')$ in $\mathbf{C}/X$ (so that $g' \circ h = g$ in $\mathbf{C}$), using the universal property of the pullback $Y \xleftarrow{f \times g} Y \xrightarrow{f' \times g'} Z \xrightarrow{g} Z'$, let $f^*h : Y \xrightarrow{f^*} Z \to Y_f \times_{Z'} Z'$ be the unique morphism making commute

$$
\begin{array}{ccc}
Y & \xrightarrow{f^*} & Z \\
\downarrow p & & \downarrow q \\
Y_f \times_{Z'} Z' & \xrightarrow{g'} & Z \\
\end{array}
$$

(Note that the outer square commutes, because $g' \circ (h \circ q) = (g' \circ h) \circ q = g \circ q = f \circ p$.) Since $p' \circ (f^*h) = p$, we get $f^*h : (Y_f \times_{Z'} Z', p) \to (Y, f \times g)$ in $\mathbf{C}/Y$, in other words $f^*h \in \mathbf{C}/Y(f^*(Z, g), f^*(Z', g'))$. So we have the action of $f^*$ on both object and morphisms. The uniqueness part of the universal property for pullbacks implies that $f^*$ respects composition $(f^*(h \circ h') = (f^*h) \circ (f^*h')$ and identities $(f^*(\text{id}(Z, g)) = \text{id}_{f^*(Z, g)})$.

(d) First note that $f_i$ acts trivially on morphisms: given $w : (W, h) \to (W', h')$ in $\mathbf{C}/Y$, since $h' \circ w = h$ in $\mathbf{C}$ we also have $(f \circ h') \circ w = f \circ h$ and hence $w : f_i(W, h) \to f_i(W', h')$. So we can define $f_i(w) \overset{\Delta}{=} w$ and in this way $f_i$ is a functor $\mathbf{C}/Y \to \mathbf{C}/X$. To see that it gives a left adjoint to $f^*$, first note that morphisms in $\mathbf{C}/X(f_i(W, h), (Z, g))$ are just morphisms $k : W \to Z$ in $\mathbf{C}$ making the outer square in (2) commute; and therefore the universal property of pullbacks says that there is a bijection

$$
\theta_{(W, h), (Z, g)} : \mathbf{C}/X(f_i(W, h), (Z, g)) \cong \mathbf{C}/Y((W, h), f^*(Z, g)).
$$

The uniqueness part of the universal property can be used to show that this bijection is natural in $(W, h)$ and $(Z, g)$ (details omitted). So we get an adjunction as required.

Question 4

(a) Given a product (3) in $\mathbf{C}$ and morphisms $y(X) \xleftarrow{\alpha} F \xrightarrow{\beta} y(Y)$ in $\text{Set}^{\mathbf{C}^{op}}$, we have to show that there is a unique morphism $\gamma : F \to y(P)$ satisfying

$$
y(\pi_1) \circ \gamma = \alpha \quad \text{and} \quad y(\pi_2) \circ \gamma = \beta \quad (6)
$$

Uniqueness of $\gamma$: If there is such a $\gamma$, then for each $\mathbf{C}$-object $Z$ and element $c \in F(Z)$, by definition of $y(\pi_1)$ and $y(\pi_2)$ we have that $\alpha_Z(c) \in \mathbf{C}(Z, X)$ and $\beta_Z(c) \in \mathbf{C}(Z, Y)$ satisfy

$$
\alpha_Z(c) = (y(\pi_1) \circ \gamma)_Z(c) = y(\pi_1)_Z(\gamma_Z(c)) = \pi_1 \circ (\gamma_Z(c))
$$

$$
\beta_Z(c) = (y(\pi_2) \circ \gamma)_Z(c) = y(\pi_2)_Z(\gamma_Z(c)) = \pi_2 \circ (\gamma_Z(c))
$$
so that \( \gamma_Z(c) = \langle \alpha_Z(c), \beta_Z(c) \rangle \in C(Z, P) \), the unique \( C \)-morphism whose compositions with \( \pi_1 \) and \( \pi_2 \) are \( \alpha_Z(c) \) and \( \beta_Z(c) \) respectively (using the universal property of the product (3)). So \( \gamma \) is uniquely determined by (6).

Existence of \( \gamma \): For each \( C \)-object \( Z \), using the universal property of the product (3), we define \( \gamma_Z : F(Z) \to C(Z, P) \) to be the function mapping each \( c \in F(Z) \) to

\[
\gamma_Z(c) = \langle \alpha_Z(c), \beta_Z(c) \rangle
\]

(7)

This gives a natural transformation \( \gamma : F \to y(P) \), since for any \( f \in C(Z', Z) \)

\[
\begin{array}{ccc}
F(Z) & \xrightarrow{\gamma_Z} & C(Z, P) = y(P)(Z) \\
\downarrow F(f) & & \downarrow f' = y(P)(f) \\
F(Z') & \xrightarrow{\gamma_{Z'}} & C(Z', P) = y(P)(Z')
\end{array}
\]

commutes in Set, because for any \( c \in F(Z) \)

\[
\gamma_{Z'}(F(f)(c)) = \langle \alpha_Z(F(f)(c)), \beta_Z(F(f)(c)) \rangle
\]

\[
= \langle \alpha_Z(c) \circ f, \beta_Z(c) \circ f \rangle \quad \text{since } \alpha \text{ and } \beta \text{ are natural transformations}
\]

\[
= \langle \alpha_Z(c), \beta_Z(c) \rangle \circ f \quad \text{by Exercise Sheet 2, question 1(a)}
\]

\[
= y(P)(f)(\gamma_Z(c)) \quad \text{by definition of the Yoneda functor } y
\]

Furthermore (6) holds since for all \( Z \in \text{obj } C \) and \( c \in (Z) \)

\[
(y(\pi_1) \circ \gamma)_Z(c) = y(\pi_1)(\gamma_Z(c)) = \pi_1 \circ (\gamma_Z(c)) \triangleq \pi_1 \circ (\langle \alpha_Z(c), \beta_Z(c) \rangle) = \alpha_Z(c)
\]

\[
(y(\pi_2) \circ \gamma)_Z(c) = y(\pi_2)(\gamma_Z(c)) = \pi_2 \circ (\gamma_Z(c)) \triangleq \pi_2 \circ (\langle \alpha_Z(c), \beta_Z(c) \rangle) = \beta_Z(c)
\]

(b) For example, consider the category \( V \) from Exercise Sheet 4, question 1, which does have all binary products. Let \( F : V \to \text{Set} \) be the functor mapping the object \( P \) to the empty set \( \emptyset \) and objects \( L \) and \( R \) to a one-element set \( 1 = \{0\} \). (Clearly this is a functor.) Note that \( F \) maps the product \( L \leftarrow P \to R \) in \( V \) to \( 1 \leftarrow \emptyset \to 1 \) (uniquely determined functions), which is not a product for \( 1 \) and \( 1 \) in \( \text{Set} \).