**2015/16 MPhil ACS / CST Part III**  
*Category Theory and Logic (L108)*  
Exercise Sheet 5 – Solution Notes

**Question 1** For each function \( f \in \text{Set}(\Sigma, \Sigma') \) there is a function \( \text{Pow} f \in \text{Set}(\text{Pow} \Sigma, \text{Pow} \Sigma') \) defined by

\[
(\text{Pow} f) S \triangleq \{ f x \mid x \in S \}
\]

Note that this is a monoid homomorphism \((\text{Pow} \Sigma, \cup, \emptyset) \to (\text{Pow} \Sigma', \cup, \emptyset)\), because

- \( \text{Pow} f \emptyset = \{ f x \mid x \in \emptyset \} = \emptyset \)
- \( \text{Pow} f (S \cup S') = \{ f x \mid x \in S \cup S' \} = \{ f x \mid x \in S \} \cup \{ f x \mid x \in S' \} = (\text{Pow} f S) \cup (\text{Pow} f S') \)

So we get \( Pf \in \text{Mon}((\text{Pow} \Sigma, \cup, \emptyset), (\text{Pow} \Sigma', \cup, \emptyset)) \). Furthermore

- \( (\text{Pow id}_\Sigma) S = \{ \text{id}_\Sigma x \mid x \in S \} = S \) for all \( S \in \text{Pow} \Sigma \); and hence \( \text{Pow id}_\Sigma = \text{id}_{\text{Pow} \Sigma} \).
- \( \text{Pow} g (\text{Pow} f S) = \{ g y \mid y \in \text{Pow} f S \} = \{ g(f x) \mid x \in S \} = \{ (g \circ f) x \mid x \in S \} = \text{Pow}(g \circ f) S \) for all \( S \in \text{Pow} \Sigma \); and hence \( P(g \circ f) = (P g) \circ (P f) \).

So \( P \) is a functor \( \text{Set} \to \text{Mon} \).

**Question 2** For each set \( \Sigma \), define \( \theta_\Sigma \in \text{Set}(\text{List} \Sigma, \text{Pow} \Sigma) \) by recursion on the length of lists:

\[
\theta_\Sigma(\text{nil}) = \emptyset \\
\theta_\Sigma(a :: \ell) = \{ a \} \cup \theta_\Sigma(\ell)
\]

Then one can prove \((\forall \ell, \ell' \in \text{List} \Sigma) \theta_\Sigma(\ell @ \ell') = \theta_\Sigma(\ell) \cup \theta_\Sigma(\ell')\) by induction on the length of \( \ell \). So we get that \( \theta_\Sigma \) is in \( \text{Mon}(\text{List} \Sigma, \text{Pow} \Sigma) \). To show that these morphisms form a natural transformation \( \theta : F \to P \), we have to show for each \( f \in \text{Set}(\Sigma, \Sigma') \) that \( Pf \circ \theta_\Sigma = \theta_\Sigma \circ F f \); and by definition of \( F \) and \( P \), this means proving \((\forall \ell \in \text{List} \Sigma) \text{Pow} f (\theta_\Sigma(\ell)) = \theta_\Sigma(\text{List} F \ell)\), which follows easily from the definitions of \( \text{Pow} f \), \( \text{List} f \) and \( \theta_\Sigma \), by induction on the length of \( \ell \).

Here is another proof, which uses the universal property of the free monoid \( F \Sigma \) instead of recursion/induction on lists.

For each set \( \Sigma \), let \( s_\Sigma \in \text{Set}(\text{List} \Sigma, \text{Pow} \Sigma) \) be the function mapping each \( x \in \Sigma \) to \( s_\Sigma(x) = \{ x \} \in \text{Pow} \Sigma \). Using the universal property of the free monoid \( i_\Sigma : \Sigma \to \text{List} \Sigma \), there is a unique monoid homomorphism \( \hat{s}_\Sigma \in \text{Mon}(F \Sigma, P \Sigma) \) with \( \hat{s}_\Sigma \circ i_\Sigma = s_\Sigma \). We take \( \theta_\Sigma \) to be \( \hat{s}_\Sigma \) and show that these functions together give a natural transformation \( \theta : F \to P \).

So we have to show for each \( f \in \text{Set}(\Sigma, \Sigma') \) that \( \theta_\Sigma \circ F f = Pf \circ \theta_\Sigma \in \text{Mon}(F \Sigma, P \Sigma') \). By the uniqueness part of the universal property of the free monoid \( i_\Sigma : \Sigma \to \text{List} \Sigma \), for this it suffices to show that the two monoid homomorphisms \( \theta_\Sigma \circ F f \) and \( Pf \circ \theta_\Sigma \), when composed with the function \( i_\Sigma \), give equal functions in \( \text{Set}(\Sigma, \text{Pow} \Sigma') \). But

\[
(P f \circ \theta_\Sigma) \circ i_\Sigma \triangleq ((\text{Pow} f) \circ s_\Sigma) \circ i_\Sigma = (\text{Pow} f) \circ (s_\Sigma \circ i_\Sigma) = (\text{Pow} f) \circ s_\Sigma \quad \text{by definition of } s_\Sigma
\]
whereas

\[(\theta_{\Sigma} \circ F f) \circ i_{\Sigma} \triangleq (s_{\Sigma} \circ F f) \circ i_{\Sigma} = s_{\Sigma} \circ (F f \circ i_{\Sigma}) = s_{\Sigma} \circ (i_{\Sigma} \circ f) \quad \text{since } i \text{ is a natural transformation}
\]

\[= (s_{\Sigma} \circ i_{\Sigma}) \circ f = s_{\Sigma} \circ f \quad \text{by definition of } s_{\Sigma}
\]

So it suffices to prove that \((\text{Pow } f) \circ s_{\Sigma} = s_{\Sigma} \circ f \in \textbf{Set}(\Sigma, \text{Pow } \Sigma)\). But for all \(x \in \Sigma\), we have \(((\text{Pow } f) \circ s_{\Sigma}) x = \text{Pow } f (s_{\Sigma} x) = \text{Pow } f \{x\} = \{f y \mid y \in \{x\}\} = \{f x\} = s_{\Sigma} (f x) = (s_{\Sigma} \circ f) x\).

**Question 3** If \(\theta \in D^C(F, G)\) is an isomorphism, then there is a natural transformation \(\theta^{-1} \in D^C(G, F)\) with \(\theta^{-1} \circ \theta = \text{id}_F\) and \(\theta \circ \theta^{-1} = \text{id}_G\). By definition of identity and composition for natural transformations, that means that for all \(X \in \text{obj } C\) we have \((\theta^{-1})_X \circ \theta_X = \text{id}_{F_X}\) and \(\theta_X \circ (\theta^{-1})_X = \text{id}_{G_X}\). Therefore each \(\theta_X \in D(F X, G X)\) is an isomorphism in \(D\) with inverse \((\theta^{-1})_X\).

Conversely, if each \(\theta_X \in D(F X, G X)\) is an isomorphism in \(D\), then the inverse morphisms \((\theta_X)^{-1}\) are natural in \(X\) because for any \(f \in C(X, Y)\) we have

\[
F f \circ (\theta_X)^{-1} = (\theta_Y)^{-1} \circ \theta_Y \circ F f \circ (\theta_X)^{-1} \quad \text{because } (\theta_Y)^{-1} \circ \theta_Y = \text{id}_{F_Y}
\]

\[= (\theta_Y)^{-1} \circ G f \circ (\theta_X)^{-1} \quad \text{because } \theta_X \text{ is natural in } X
\]

\[= (\theta_Y)^{-1} \circ G f \circ (\theta_X)^{-1} \quad \text{because } \theta_X \circ (\theta^{-1})_X = \text{id}_{G_X}
\]

and so determine a natural transformation \(\theta \in D^C(G, F)\) with \((\theta^{-1})_X \triangleq (\theta_X)^{-1}\) for each \(X \in \text{obj } C\). This gives an inverse for \(\theta\). For \((\theta^{-1})_X \circ \theta_X = (\theta^{-1})_X \circ \theta_X = (\theta_X)^{-1} \circ \theta_X = \text{id}_{F_X} = (\text{id}_F)_X\), so that \(\theta^{-1} \circ \theta = \text{id}_F\); and similarly, \(\theta \circ \theta^{-1} = \text{id}_G\).

**Question 4** If \(ch_X\) were natural in \(X\), then taking \(X = 2 = \{0, 1\}\) and letting \(\tau\) be as in the hint, there would be a commutative square in \(\textbf{Set}\):

\[
\begin{array}{c}
\begin{array}{c}
P^+2 \\
P^+\tau
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\downarrow ch_2
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
P^+2 \\
P^+\tau
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \tau
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\]

Consider \(\{0, 1\} \in P^+2\). We have

\[
P^+\tau \{0, 1\} = \{\tau 0, \tau 1\} = \{1, 0\} = \{0, 1\}
\]

(4)

Since \(ch_2(\{0, 1\}) \in \{0, 1\}\), either \(ch_2(\{0, 1\}) = 0\), or \(ch_2(\{0, 1\}) = 1\). In the first case we get

\[
1 = \tau 0 = \tau(ch_2\{0, 1\}) = ch_2(P^+\tau \{0, 1\}) \quad \text{by (3)}
\]

\[
= ch_2\{0, 1\} \quad \text{by (4)}
\]

\[
= 0 \quad \text{by assumption}
\]

which is a contradiction; and in the second case we get a similar contradiction. So (3) cannot commute and in particular \(ch_X\) cannot be natural in \(X\).
Question 5

(a) Define \((I \alpha)_X \triangleq I(\alpha_X) : I(FX) \to I(GX)\). Since \(\alpha_X\) is natural in \(X \in \text{obj } C\), we have \(Gf \circ \alpha_X = \alpha_Y \circ Ff\); and then since \(I\) is a functor, we get \(I(Gf) \circ I(\alpha_X) = I(\alpha_Y) \circ I(Ff)\). So \((I \alpha)_X\) is natural in \(X\).

(b) Define \((\gamma_f)_X \triangleq \gamma_{FX} : I(FX) \to I(FX)\). Since \(\gamma_Y\) is natural in \(Y \in \text{obj } D\), \((\gamma_f)_X\) is natural in \(X \in \text{obj } C\).

(c) Define \((\beta \circ \alpha)_X \triangleq \beta_X \circ \alpha_X : FX \to HX\). Since \(\alpha_X\) and \(\beta_X\) are natural in \(X \in \text{obj } C\), so is \((\beta \circ \alpha)_X\).

(d) Define \((\gamma \circ \alpha)_X \triangleq \gamma_{GX} \circ I(\alpha_X) : I(FX) \to I(GX)\). This is natural in \(X\), because for any \(f \in C(X,Y)\)

\[
J(Gf) \circ (\gamma \circ \alpha)_X \triangleq J(Gf) \circ \gamma_{GX} \circ I(\alpha_X)
\]

\[
= J(Gf) \circ I(\alpha_X) \circ \gamma_{FX} \quad \text{by naturality for } \gamma
\]

\[
= J(Gf \circ \alpha_X) \circ \gamma_{FX} \quad \text{by functoriality for } J
\]

\[
= J(\alpha_Y \circ Ff) \circ \gamma_{FX} \quad \text{by naturality for } \alpha
\]

\[
= J(\alpha_Y) \circ J(Ff) \circ \gamma_{FX} \quad \text{by functoriality for } J
\]

\[
= J(\alpha_Y) \circ \gamma_{FY} \circ I(Ff) \quad \text{by naturality for } \gamma
\]

\[
= \gamma_{GY} \circ I(\alpha_Y) \circ I(Ff) \quad \text{by naturality for } \gamma
\]

\[
\triangleq (\gamma \circ \alpha)_Y \circ I(Ff)
\]

(e) \(((\delta \circ \beta) \circ (\gamma \circ \alpha))_X \triangleq (\delta \circ \beta)_X \circ (\gamma \circ \alpha)_X
\]

\[
\triangleq \delta_{HX} \circ J(\beta_X) \circ \gamma_{GX} \circ I(\alpha_X)
\]

\[
= \delta_{HX} \circ \gamma_{HX} \circ I(\beta_X) \circ I(\alpha_X) \quad \text{by naturality for } \gamma
\]

\[
\triangleq (\delta \circ \gamma)_{HX} \circ I(\beta_X) \circ I(\alpha_X)
\]

\[
= (\delta \circ \gamma)_H \circ I(\beta_X \circ \alpha_X) \quad \text{by functoriality for } I
\]

\[
\triangleq (\delta \circ \gamma)_H \circ I((\beta \circ \alpha)_X)
\]

\[
\triangleq (\delta \circ \gamma)(\beta \circ \alpha)_X
\]

Question 6

(a) We use the notation \(\overline{g} \triangleq \theta_{XY}(g)\) and \(\overline{f} \triangleq \theta^{-1}_{XY}(f)\) from Lecture 14.

Define \(\eta_X \triangleq \overline{id_{FX}} \in C(X, G(FX))\). This is natural in \(X \in \text{obj } C\), because using naturality for \(\theta\) (twice) we have

\[
G(Ff) \circ \eta_X \triangleq G(Ff) \circ \overline{id_{FX}} = \overline{Ff} \circ \overline{id_{FX}} = \overline{id_{FY} \circ Ff} = \overline{id_{FY} \circ f} \triangleq \eta_Y \circ f
\]

Dually, define \(\epsilon_Y \triangleq \overline{id_{GY}} \in D(F(GY), Y)\) and prove it is natural in \(Y \in \text{obj } D\) by a similar calculation.
(b) \((\varepsilon_F \circ F \eta)_X \triangleq (\varepsilon_F)_X \circ (F \eta)_X\)
\[\triangleq \varepsilon_{FX} \circ F(\eta_X)\]
\[\triangleq \text{id}_{G(FX)} \circ F(\eta_X)\]
\[= \text{id}_{G(FX)} \circ \eta_X\quad \text{by naturality of } \theta\]
\[= \eta_X\]
\[= \text{id}_{FX}\quad \text{since } \theta \text{ is an isomorphism}\]
\[\triangleq (\text{id}_F)_X\]

The proof that \((G \varepsilon \circ \eta_G)_Y = (\text{id}_G)_Y\) is dual to the one above.

**Question 7**  This is a standard result; see for example Proposition 10.1 on page 254 of Awodey’s *Category Theory* book.