2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 4 – Solution Notes

Question 1

(a)
\[ V(L, L) = \{\text{id}_L\} \quad V(P, L) = \{p\} \quad V(R, L) = \emptyset \]
\[ V(L, P) = \emptyset \quad V(P, P) = \{\text{id}_P\} \quad V(R, P) = \emptyset \]
\[ V(L, R) = \emptyset \quad V(P, R) = \{q\} \quad V(R, R) = \{\text{id}_R\} \]

By inspection, if two \( V \)-morphisms \( f \) and \( g \) can be composed, that is, satisfy \( \text{cod} f = \text{dom} g \), then either \( f \) is an identity morphism and the composition \( g \circ f \) has to be \( g \), or \( g \) is an identity morphism and the composition \( g \circ f \) has to be \( f \). So composition is uniquely determined, given the above sets of morphisms.

(b) \( V \) does not have a terminal object (\( L \) and \( P \) are not terminal because there are no morphisms to them from \( R \); \( R \) is not terminal, because there is no morphism to it from \( L \)). \( V^{\text{op}} \) does have a terminal object, namely \( P \) (there is a unique morphism from \( P \) to each of \( L, P \) and \( R \) in \( V \) – so it is initial in \( V \) and hence terminal in \( V^{\text{op}} \)).

(c) Note that \( V \) (and hence also \( V^{\text{op}} \)) is a category arising from a poset. Furthermore in this poset every pair of elements has a greatest lower bound (in particular the glb of \( L \) and \( R \) is \( P \)). So \( V \) has binary products. However, \( V^{\text{op}} \) does not have them, since in the poset \( V^{\text{op}} \), \( \{L, R\} \) has no lower bound, let alone a greatest one.

Question 2

(a) For the eight choices of \((x, y, z)\) in \( \Sigma \times \Sigma \times \Sigma \) one can check that \((x \oplus y) \oplus z = x \oplus (y \oplus z) \) and \((x \otimes y) \otimes z = x \otimes (y \otimes z) \). So the two binary operations are associative. Furthermore \( e_M \triangleq b \) is a unit for \( \oplus \) and \( e_N \triangleq a \) is a unit for \( \otimes \). So \( M \triangleq (\Sigma, \oplus, b) \) and \( N \triangleq (\Sigma, \otimes, a) \) are monoids.

(b) \( M \) and \( N \) are not isomorphic in \( \text{Mon} \). For if there was an isomorphism \( i : M \cong N \), then since \( i \) is in particular a monoid homomorphism, we would have \( i(a \oplus a) = i(a) \times i(a) \) and \( i(e_M) = e_N \), that is, \( i(b) = a \); and since monoid isomorphisms are in particular bijective functions, the latter implies that we must also have \( i(a) = b \). Hence \( b = i(a) = i(a \oplus a) = i(a) \times i(a) = b \times b = a \), contradicting the fact that \( a \neq b \). So no such isomorphism \( i \) can exist.

Question 3

(a) For \( i = 1, 2 \) we have
\[ \pi_i \circ (\delta_X \circ f) = \pi_i \circ (\text{id}_Y, \text{id}_Y) \circ f = \text{id}_Y \circ f = f \]
and
\[
\pi_i \circ ((f \times f) \circ \delta_X) = \pi_i \circ (f \circ \pi_1, f \circ \pi_2) \circ \delta_X = f \circ \pi_i \circ \delta_X = f \circ \text{id}_X = f
\]
and therefore \(\delta_X \circ f = (f \times f) \circ \delta_X\), by the uniqueness part of the universal property of the product \(Y \leftarrow \pi_1 Y \times Y \pi_2 \rightarrow Y\).

(b) We have
\[
\pi_1 \circ (\tau_X \circ \delta_X) = \pi_1 \circ (\pi_2, \pi_1) \circ \delta_X = \pi_2 \circ \delta_X = \text{id}_X
\]
and similarly \(\pi_2 \circ (\tau_X \circ \delta_X) = \text{id}_X\). Therefore \(\tau_X \circ \delta_X = (\text{id}_X, \text{id}_X) = \delta_X\), by the uniqueness part of the universal property of the product \(X \leftarrow \pi_1 X \times X \pi_2 \rightarrow X\).

(c) We have
\[
\pi_1 \circ (\tau_X \circ \tau_X) = \pi_1 \circ (\pi_2, \pi_1) \circ \tau_X = \pi_2 \circ \tau_X = \pi_1
\]
and similarly \(\pi_2 \circ (\tau_X \circ \tau_X) = \pi_2\). Therefore \(\tau_X \circ \tau_X = (\pi_1, \pi_2) = \pi_{X \times X}\), by the uniqueness part of the universal property of the product \(X \leftarrow \pi_1 X \times X \pi_2 \rightarrow X\).

**Question 4**

(a) Given \(k_1, k_2 : Z \rightarrow X\) with \(e \circ k_1 = e \circ k_2\), we have to show \(k_1 = k_2\). Putting \(h \triangleq e \circ k_1 = e \circ k_2\), we have \(f \circ h = (f \circ e) \circ k_1 = (g \circ e) \circ k_1 = g \circ h\) and \(e \circ k = h\) for both \(k = k_1\) and \(k = k_2\); so by the uniqueness part of the property of being an equalizer, \(k_1 = k_2\).

(b) The morphism \(f\) has equal compositions with both \(f \circ g\) and \(\text{id}_Y\), since \((f \circ g) \circ f = f \circ \text{id}_X = f = \text{id}_Y \circ f\). If for some \(h\) we have \((f \circ g) \circ h = \text{id}_Y \circ h\), then \(h = f \circ (g \circ h)\); and \(g \circ h\) is the unique such morphism, because if \(k : Z \rightarrow X\) also satisfies \(h = f \circ k\), then \(k = \text{id}_X \circ k = (g \circ f) \circ k = g \circ (f \circ k) = g \circ h\).

(c) The equalizer of \(f, g \in \text{Set}(X, Y)\) is the inclusion \(e : E \triangleq \{x \in X \mid f x = g x\} \hookrightarrow X\); in other words \(e \in \text{Set}(E, X)\) is the function \(\{(x, x) \mid x \in E\}\).

For if \(h \in \text{Set}(Z, X)\) satisfies \(f \circ h = g \circ h\), then for all \(z \in Z\), \(h z \in E\); so \(h\) factors through the inclusion \(e : E \hookrightarrow X\), that is \(h = e \circ k\), where \(k \in \text{Set}(Z, E)\) is the function \(\{(z, h z) \mid z \in Z\}\); and it does so uniquely because inclusions, being injective functions, are monomorphisms in \(\text{Set}\).

**Question 5**

(a) \((X, \text{id}_X)\) is a terminal object in \(\text{C}/X\), because for any object \((A, p)\) we have
\[
p \in \text{C}/X((A, p),(X, \text{id}_X))
\]
(since \(\text{id}_X \circ p = p\)); and for any \(q \in \text{C}/X((A, p),(X, \text{id}_X))\) we have \(\text{id}_X \circ q = p\) (by definition of morphisms in \(\text{C}/X\)) and hence \(q = p\).

(b) The product of \((A, p)\) and \((B, q)\) in \(\text{Set}/X\) is
\[
(A, p) \leftarrow \pi_1 (P, r) \pi_2 \rightarrow (B, q)
\]
Therefore there can be no such pure term to the usual order relation. Recall that in this ccc, the terminal object cartesian closed category satisfying how the ground types involved are mapped to objects in the ccc. If there were a pure term sections. Since a pure term contains no constants, its meaning in the ccc only depends on interpreting an interpretation function.

**Question 6**

(a) The product of $X$ and $Y$ in $\mathbf{C}$ is their coproduct in $\mathbf{Set}$, which is the disjoint union

$$X \uplus Y = \{(x,0) \mid x \in X\} \cup \{(y,1) \mid y \in Y\}$$

together with the functions $\text{inl} \in \text{Set}(X, X \uplus Y)$ and $\text{inr} \in \text{Set}(Y, X \uplus Y)$ that respectively map $x \in X$ to $(x,0) \in X \uplus Y$ and $y \in Y$ to $(y,1) \in X \uplus Y$.

(b) Consider the one-element set $1 = \{0\}$ as an object of $\mathbf{C}$. If the exponential $1^1$ existed in $\mathbf{C}$, there would be a bijection $\mathbf{C}(1 \times 1, 1) \cong \mathbf{C}(1, 1^1)$. But from part (a)

$$\mathbf{C}(1 \times 1, 1) \cong \mathbf{Set}(1, 1 \uplus 1)$$

is a two-element set, whereas

$$\mathbf{C}(1, 1^1) \cong \mathbf{Set}(1^1, 1)$$

has exactly one element no matter what set $1^1$ is. Thus for any set $X$, the sets $\mathbf{C}(1 \times 1, 1)$ and $\mathbf{C}(1, X)$ cannot be in bijection and therefore the exponential $1^1$ of 1 and 1 in $\mathbf{C}$ cannot exist.

**Question 7** Recall that the semantics of STLC types and terms in a ccc depends upon giving an interpretation function $M$ mapping ground types to objects and constants to global sections. Since a pure term contains no constants, its meaning in the ccc only depends on how the ground types involved are mapped to objects in the ccc. If there were a pure term $t$ satisfying $\vdash t : ((\mathbf{G} \to \mathbf{G'}) \to \mathbf{G}) \to \mathbf{G}$, then for any interpretation $M$ of the ground types in a cartesian closed category $\mathbf{C}$, we would get a morphism

$$M[\vdash t : ((\mathbf{G} \to \mathbf{G'}) \to \mathbf{G}) \to \mathbf{G}] : \mathbf{C}(\mathcal{T}, \mathbf{X}(\mathbf{Y}(\mathbf{X})))$$

where $X = M(\mathbf{G})$ and $Y = M(\mathbf{G'})$.

But consider when $\mathbf{C}$ is the cartesian closed preorder given by the unit interval $[0,1]$ with the usual order relation. Recall that in this ccc, the terminal object $\top$ is $1 \in [0,1]$; and given $X, Y \in [0,1]$ their exponential (Heyting implication) $Y^X$ is

$$Y^X = \begin{cases} 1 & \text{if } X \leq Y \\ Y & \text{otherwise} \end{cases}$$

If (3) holds in this $\mathbf{C}$, then $1 \leq \mathbf{X}(\mathbf{Y}(\mathbf{X}))$, that is $\mathbf{X}(\mathbf{Y}(\mathbf{X})) = 1$. But we can take $M$ to map $\mathbf{G}$ to $\frac{1}{2}$ and $\mathbf{G'}$ to 0, in which case we get $Y^X = 0$, so $\mathbf{X}(\mathbf{Y}(\mathbf{X})) = 1$ and hence $\mathbf{X}(\mathbf{Y}(\mathbf{X})) = \frac{1}{2} \neq 1$. Therefore there can be no such pure term $t$. 

where $P \cong \{(a,b) \in A \times B \mid p \; a = q \; b\}$ and for all $(a,b) \in P$

$$r(a,b) \cong p \; a = q \; b \quad \pi_1(a,b) \cong a \quad \pi_2(a,b) \cong b$$

For if we have $(A,p) \not\rightarrow (Y,s) \rightarrow (B,q)$ in $\mathbf{Set}/X$, then $(f,g) : Y \to A \times B$ factors through the subset $P \subseteq A \times B$ (since for all $y \in Y$, $p(f y) = s \; y = q(g \; y)$) to give a morphism $(f,g) : (Y,s) \to (P,r)$ with $\pi_1 \circ (f,g) = f$ and $\pi_2 \circ (f,g) = g$. It is unique with this property, since if $h : (Y,s) \to (P,r)$ also satisfies $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$, then for all $y \in Y$, $h \; y = (f \; y,g \; y) = (f,g) \; y$, so that $h = (f,g)$. 

$$X \uplus Y = \{(x,0) \mid x \in X\} \cup \{(y,1) \mid y \in Y\}$$