1. Let $C$ be a category with binary products.
   
   (a) For morphisms $f \in C(X,Y)$, $g_1 \in C(Y,Z_1)$ and $g_2 \in C(Y,Z_2)$, show that
   \[
   (g_1, g_2) \circ f = (g_1 \circ f, g_2 \circ f) \in C(X, Z_1 \times Z_2) \tag{1}
   \]
   
   (b) For morphisms $f_1 \in C(X_1,Y_1)$ and $f_2 \in C(X_2,Y_2)$, define
   \[
   f_1 \times f_2 \triangleq (f_1 \circ \pi_1, f_2 \circ \pi_2) \in C(X_1 \times X_2, Y_1 \times Y_2) \tag{2}
   \]

   For any $g_1 \in C(Z,X_1)$ and $g_2 \in C(Z,X_2)$, show that
   \[
   (f_1 \times f_2) \circ (g_1, g_2) = (f_1 \circ g_1, f_2 \circ g_2) \in C(Z, Y_1 \times Y_2) \tag{3}
   \]
   
   (c) Show that the operation $f_1, f_2 \mapsto f_1 \times f_2$ defined in part (1b) satisfies
   \[
   (h_1 \times h_2) \circ (k_1 \times k_2) = (h_1 \circ k_1) \times (h_2 \circ k_2) \tag{4}
   \]
   \[
   \text{id}_X \times \text{id}_Y = \text{id}_{X \times Y} \tag{5}
   \]

2. Let $C$ be a category with binary products ($\_ \times \_ \_)$ and a terminal object ($\top$). Given objects $X, Y, Z \in C$, construct isomorphisms
   \[
   \alpha_{X,Y,Z} : X \times (Y \times Z) \cong (X \times Y) \times Z \tag{6}
   \]
   \[
   \lambda_X : \top \times X \cong X \tag{7}
   \]
   \[
   \rho_X : X \times \top \cong X \tag{8}
   \]
   \[
   \tau_{X,Y} : X \times Y \cong Y \times X \tag{9}
   \]

3. A pairing for a monoid $(M, \cdot, e)$ consists of elements $p_1, p_2 \in M$ and a binary operation
   \[
   \langle \_, \_ \rangle : M \times M \to M \text{ satisfying for all } x, y, z \in M
   \]
   \[
   p_1 \cdot \langle x, y \rangle = x \tag{10}
   \]
   \[
   p_2 \cdot \langle x, y \rangle = y \tag{11}
   \]
   \[
   \langle p_1, p_2 \rangle = e \tag{12}
   \]
   \[
   \langle x, y \rangle \cdot z = \langle x \cdot z, y \cdot z \rangle \tag{13}
   \]

   Given such a pairing, show that the monoid, when regarded as a one-object category, has binary products.

4. A monoid $(M, \cdot_M, e_M)$ is said to be abelian if its multiplication is commutative: $(\forall x, y \in M) x \cdot_M y = y \cdot_M x.$
(a) If \((M, \cdot_M, e_M)\) is an abelian monoid, show that the functions \(m \in \text{Set}(M \times M, M)\) and \(u \in \text{Set}(\top, M)\) defined by
\[
m(x, y) = x \cdot_M y \quad \text{(all } x, y \in M) \\
u(0) = e_M
\]
determine morphisms in the category \(\text{Mon}\) of monoids, \(m \in \text{Mon}(M \times M, M)\) and \(u \in \text{Mon}(\top, M)\) (where as usual we just write \(M\) for the monoid \((M, \cdot_M, e_M)\) and \(\top\) for the terminal monoid \((\top, \cdot_\top, e_\top)\) with \(\top\) a one-element set, \(\{0\}\) say, \(0 \cdot_\top 0 = 0\) and \(e_\top = 0\)).
Show further that \(m\) and \(u\) make the monoid \(M\) into a “monoid object in the category \(\text{Mon}\)”, in the sense that the following diagrams in \(\text{Mon}\) commute:

\[
\begin{array}{c}
(M \times M) \times M \xrightarrow{m \times \text{id}} M \\
\cong \quad \cong \\
\text{id} \times M \\
\downarrow \quad \downarrow \\
M \times (M \times M) \xrightarrow{m} M \\
\end{array}
\quad \text{(associativity)} \quad (14)
\]

\[
\begin{array}{c}
\top \times M \xrightarrow{u \times \text{id}} M \\
\pi_2 \cong \text{id} \\
M \xrightarrow{\text{id}} M \\
\end{array}
\quad \text{(left unit)} \quad (15)
\]

\[
\begin{array}{c}
M \times \top \xrightarrow{\text{id} \times u} M \\
\pi_1 \cong \text{id} \\
M \xrightarrow{\text{id}} M \\
\end{array}
\quad \text{(right unit)} \quad (16)
\]

(b) Show that every monoid object in the category \(\text{Mon}\) (in the above sense) arises as in (4a). [Hint: if necessary, search the internet for “Eckmann-Hilton argument”.

5. Let \(\text{AbMon}\) be the category whose objects are abelian monoids (question 4) and whose morphisms, identity morphisms and composition are as in \(\text{Mon}\).

(a) Show that the product in \(\text{Mon}\) of two abelian monoids gives their product in \(\text{AbMon}\).

(b) Given \(M, N \in \text{AbMon}\) define morphisms \(i \in \text{AbMon}(M, M \times N)\) and \(j \in \text{AbMon}(N, M \times N)\) that make \(M \times N\) into a coproduct in \(\text{AbMon}\).

6. The category \(\text{Set}^\top\) of ‘sets evolving through discrete time’ is defined as follows:

- Objects are triples \((X, (\_)^+, |\_|)\), where \(X \in \text{Set}\), \((\_)^+ \in \text{Set}(X, X)\) and \(|\_| \in \text{Set}(X, \mathbb{N})\) satisfy for all \(x \in X\)
  \[
  |x^+| = |x| + 1
  \quad (17)
  \]
  [Think of \( |x| \) as the instant of time at which \(x\) exists and \(x \mapsto x^+\) as saying how an element evolves from one instant to the next.]
• Morphisms \( f : (X, (\_^+, |\_|)) \to (Y, (\_^+, |\_|)) \) are functions \( f \in \textbf{Set}(X, Y) \) satisfying for all \( x \in X \)

\[
\begin{align*}
(f x)^+ &= f(x^+) \\
|fx| &= |x|
\end{align*}
\]  

(18)  

(19)

• Composition and identities are as in the category \textbf{Set}.

Show that \( \text{Set}^w \) has a terminal object and binary products.

7. Show that the category \textbf{Pre} of pre-ordered sets and monotone functions is a cartesian closed category.