1. (a) Show that the sets $2 = \{0, 1\}$ and $3 = \{0, 1, 2\}$ are not isomorphic in the category $\text{Set}$ of sets and functions.

(b) Let $P$ be the pre-ordered set with underlying set $\{0, 1\}$ and pre-order: $0 \leq 0$, $1 \leq 1$. Let $Q$ be the pre-ordered set with the same underlying set and pre-order: $0 \leq 0$, $0 \leq 1$, $1 \leq 1$. Show that $P$ and $Q$ are not isomorphic in the category $\text{Pre}$ of pre-ordered sets and monotone functions.

(c) Why are the sets $\mathbb{N}$ (natural numbers) and $\mathbb{Q}$ (rational numbers) isomorphic in $\text{Set}$? Regarding them as pre-ordered sets via the usual ordering on numbers, show that they are not isomorphic in $\text{Pre}$.

2. Let $C$ be a category and let $f \in C(X, Y)$ and $g \in C(Y, Z)$ be morphisms in $C$.

(a) Prove that if $f$ and $g$ are both isomorphisms, then so is $g \circ f$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(b) Prove that if $f$ and $g \circ f$ are both isomorphisms, then so is $g$.

(c) If $g \circ f$ is an isomorphism, does that necessarily imply that either of $f$ or $g$ are isomorphisms?

3. Let $\text{Mat}$ be a category whose objects are all the non-zero natural numbers $1, 2, 3, \ldots$ and whose morphisms $M \in \text{Mat}(m, n)$ are $m \times n$ matrices with real number entries. If composition is given by matrix multiplication, what are the identity morphisms? Give an example of an isomorphism in $\text{Mat}$ that is not an identity. Can two object $m$ and $n$ be isomorphic in $\text{Mat}$ if $m \neq n$?

4. Let $C$ be a category. A morphism $f : X \to Y$ in $C$ is called a monomorphism, if for every object $Z \in C$ and every pair of morphisms $g, h : Z \to X$ we have

$$f \circ g = f \circ h \Rightarrow g = h$$

It is called a split monomorphism if there is some morphism $g : Y \to X$ with $g \circ f = \text{id}_X$, in which case we say that $g$ is a left inverse for $f$.

(a) Prove that every isomorphism is a split monomorphism and that every split monomorphism is a monomorphism.

(b) Prove that if $f : X \to Y$ and $g : Y \to Z$ are monomorphisms, then $g \circ f : X \to Z$ is a monomorphism.

(c) Prove that if $f : X \to Y$ and $g : Y \to Z$ are morphisms in $C$, and $g \circ f$ is a monomorphism, then $f$ is a monomorphism.

(d) Characterize the monomorphisms in the category $\text{Set}$ of sets and functions. Is every monomorphism in $\text{Set}$ a split monomorphism?

(e) By considering the category $\text{Set}$, show that a split monomorphism can have more than one left inverse.
(f) Regarding a pre-ordered set \((P, \leq)\) as a category, which of its morphisms are monomorphisms and which are split monomorphisms?

5. The dual of monomorphism is called epimorphism: a morphism \(f : X \to Y\) in \(\mathbf{C}\) is an epimorphism iff \(f \in \mathbf{C}^{\text{op}}(Y, X)\) is a monomorphism in \(\mathbf{C}^{\text{op}}\).

(a) Show that \(f \in \mathbf{Set}(X, Y)\) is an epimorphism iff \(f\) is a surjective function.

(b) Regarding a pre-ordered set \((P, \leq)\) as a category, which of its morphisms are epimorphisms?

(c) Give an example of a category containing a morphism that is both an epimorphism and a monomorphism, but not an isomorphism. [Hint: consider your answers to (4f) and (5b).]

6. Let \(\mathbf{C}\) be the category the following category:

- \(\mathbf{C}\)-objects are triples \((X, x_0, x_s)\) where \(X \in \mathbf{Set}\), \(x_0 \in X\) and \(x_s \in \mathbf{Set}(X, X)\);
- \(\mathbf{C}\)-morphisms \(f \in \mathbf{C}((X, x_0, x_s), (Y, y_0, y_s))\) are functions \(f \in \mathbf{Set}(X, Y)\) satisfying \(f x_0 = y_0\) and \(f \circ x_s = y_s \circ f\);
- composition and identities are as for the category \(\mathbf{Set}\).

(a) Show that \(\mathbf{C}\) has a terminal object.

(b) Show that \(\mathbf{C}\) has an initial object whose underlying set is the set \(\mathbb{N} = \{0, 1, 2, 3, \ldots\}\) of natural numbers.

7. In a category \(\mathbf{C}\) with a terminal object 1, a morphism \(p : 1 \to X\) is called a point (or global element) of the object \(X\). \(\mathbf{C}\) is said to be well-pointed if for all objects \(X, Y \in \mathbf{C}\), two morphisms \(f, g : X \to Y\) are equal if their compositions with all points of \(X\) are equal:

\[ ((\forall p \in \mathbf{C}(1, X)) f \circ p = g \circ p) \Rightarrow f = g \]  \hspace{1cm} (1)

(a) Show that \(\mathbf{Set}\) is well-pointed.

(b) Is the opposite category \(\mathbf{Set}^{\text{op}}\) well-pointed? [Hint: observe that the left-hand side of the implication in (1) is vacuously true in the case that \(\mathbf{C}(1, X)\) is empty.]