2016/17 MPhil ACS / CST Part III
Category Theory and Logic (L108)
Exercise Sheet 1 – Solution Notes

Question 1

(a) In Lecture 2 we saw that a morphism in \( \textbf{Set} \) is an isomorphism iff it is a bijection; but there is no bijection \( 3 \cong 2 \), since any function \( f : 3 \to 2 \) cannot be injective.

(b) Any function \( f : Q \to P \) that is monotonic satisfies \( f 0 \leq f 1 \) in \( P \) and hence \( f 0 = f 1 \). So \( f \) is not a bijection. But any isomorphism in \( \textbf{Pre} \) is in particular an isomorphism in \( \textbf{Set} \) of the underlying sets (why?) and hence a bijection.

(c) Recall that the set \( Q \) of rational numbers is countably infinite, that is, in bijection with \( \mathbb{N} \); so \( \mathbb{N} \) and \( Q \) are isomorphic in \( \textbf{Set} \). However, as a pre-ordered set the rationals are dense: writing \( x < y \) to mean \( x \leq y \land x \neq y \), we have \( (\forall x,y \in Q) \ x < y \Rightarrow (\exists z \in Q) \ x < z \land z < y \); whereas \( (\mathbb{N}, \leq) \) is not a dense pre-ordered set. It is not hard to see that the density property of pre-ordered sets is preserved under isomorphism. So \( (\mathbb{N}, \leq) \) cannot be isomorphic to \( (Q, \leq) \) in \( \textbf{Pre} \).

Question 2

(a)

\[
(g \circ f) \circ (f^{-1} \circ g^{-1}) = (g \circ (f \circ f^{-1})) \circ g^{-1} \quad \text{(associativity)}
\]

\[
= (g \circ \text{id}_Y) \circ g^{-1} \quad \text{(definition of } f^{-1})
\]

\[
= g \circ g^{-1} \quad \text{(unity)}
\]

\[
= \text{id}_Z \quad \text{(definition of } g^{-1})
\]

and a similar proof shows that \( (f^{-1} \circ g^{-1}) \circ (g \circ f) = \text{id}_X \). So \( g \circ f \) is an isomorphism with inverse \( f^{-1} \circ g^{-1} \).

(b) If \( f \) and \( g \circ f \) have inverses \( f^{-1} \in \textbf{C}(Y, X) \) and \( (g \circ f)^{-1} \in \textbf{C}(Z, X) \), then consider \( h \triangleq f \circ (g \circ f)^{-1} \in \textbf{C}(Z, Y) \). We have

\[
g \circ h = g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} = \text{id}_Z
\]

and

\[
h \circ g = (f \circ (g \circ f)^{-1}) \circ g = f \circ (g \circ f)^{-1} \circ g \circ f \circ f^{-1} = f \circ f^{-1} = \text{id}_Y
\]

so that \( g \) is an isomorphism with inverse \( h \).

(c) No. In the category \( \textbf{Set} \) take \( X = \{0\} = Z, Y = \{0,1\}, f \in \textbf{Set}(X, Y) \) to be the function \( f 0 = 0 \) and \( g \in \textbf{Set}(Y, Z) \) to be the function with constant value 0. Then neither \( f \) nor \( g \) are isomorphisms (since they are not bijections), but \( g \circ f = \text{id}_X \) is one.
Question 3  The identity morphism $\text{id}_n \in \text{Mat}(n,n)$ is the $n \times n$ matrix whose $(i,j)^{th}$ entry is 1 if $i = j$ and is 0 otherwise.

The morphism $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{Mat}(2,2)$ is a non-identity isomorphism since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \text{id}_2$.

Two objects $m$ and $n$ are isomorphic in $\text{Mat}$ only if $m = n$. For if $M \in \text{Mat}(m,n)$ is an isomorphism, then $M = \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_m \end{pmatrix}$ consists of $m$ rows that are linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$: for if $\lambda_1 \vec{v}_1 + \cdots + \lambda_m \vec{v}_m = \vec{0} \in \mathbb{R}^n$, that is, $(\lambda_1 \cdots \lambda_m) M = \vec{0}$, then applying the inverse of $M$ we get $\vec{0} = \vec{0} M^{-1} = (\lambda_1 \cdots \lambda_m) MM^{-1} = (\lambda_1 \cdots \lambda_m)$. So since $\mathbb{R}^n$ is a vector space of dimension $n$, we must have $m \leq n$. By a symmetric argument, $n \leq m$.

Question 4

(a) If $f$ is an isomorphism, its inverse $f^{-1}$ is in particular a left inverse. If $g$ is a left inverse for $f$, then for all $h, k \in \text{C}(Z, X)$ we have $f \circ h = f \circ k \Rightarrow h = \text{id}_X \circ h = g \circ f \circ h = g \circ f \circ k = \text{id}_X \circ k = k$, so that $f$ is a monomorphism.

(b) If $h, k \in \text{C}(W, X)$ satisfy $(g \circ f) \circ h = (g \circ f) \circ k$, then $f \circ h = f \circ k$ since $g$ is a monomorphism; and then $h = k$ since $f$ is a monomorphism.

(c) If $h, k \in \text{C}(W, X)$ satisfy $f \circ h = f \circ k$, then $(g \circ f) \circ h = (g \circ f) \circ k$ and since $g \circ f$ is a monomorphism, this implies $h = k$.

(d) The monomorphisms in $\text{Set}$ are exactly the injective functions.

Proof. If $f \in \text{Set}(X, Y)$ is injective, then for any $g, h \in \text{Set}(Z, X)$, if $f \circ g = f \circ h$, then for all $z \in Z$ we have $f(g(z)) = f(h(z))$, so $g(z) = h(z)$ (since $f$ is injective); therefore $g$ and $h$ are equal functions.

Conversely, if $f \in \text{Set}(X, Y)$ is a monomorphism, then for any $x, x' \in X$ let $\gamma x, \gamma x' \in \text{Set}(1, X)$ be the functions mapping the unique element of 1 = $\{0\}$ to $x$ and $x'$ respectively. If $f x = f x'$, then $f \circ \gamma x = f \circ \gamma x' \in \text{Set}(1, Y)$. Since $f$ is a monomorphism, this implies $\gamma x = \gamma x'$ and hence $x = \gamma x 0 = \gamma x' 0 = x'$. So $f$ is injective.

Not every monomorphism in $\text{Set}$ is split. For example, consider the unique morphism in $\text{Set}(\emptyset, 1)$ (where $\emptyset$ denotes the empty set). This is injective (vacuously), but there is no function $1 \to \emptyset$ in $\text{Set}$.

(e) Consider $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$ and the injective function $f \in \text{Set}(2, 3)$ with $f 0 = 0$ and $f 1 = 1$. There are two different left inverses for $f$, one mapping 2 to 0 and the other mapping 2 to 1.

(f) All morphisms in a pre-ordered set are monomorphisms, because there is at most one morphism between two objects. The only split monomorphisms are the isomorphisms (since if $f : p \to q$ and $g : q \to p$ then $f$ and $g$ are isomorphisms, since $g \circ f$ and
\(f \circ g\) are necessarily equal to the unique morphism, namely the identity, on \(p\) and \(q\) respectively.

**Question 5**

(a) Suppose \(f \in \text{Set}(X, Y)\) is surjective. If \(g, h \in \text{Set}(Y, Z)\) and \(g \circ f = h \circ f\), then for all \(y \in Y\), there exists \(x \in X\) with \(y = f(x)\) (since \(f\) is surjective) and hence \(g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)\); therefore \(g\) and \(h\) are equal functions.

Conversely, suppose \(f \in \text{Set}(X, Y)\) is an epimorphism. For each \(y \in Y\), consider the functions \(g_y, h_y \in \text{Set}(Y, \{0, 1\})\) that map \(y\) to 0 and to 1 respectively, and map all other elements of \(Y\) to 0. Since \(g_y \neq h_y\) and \(f\) is an epimorphism, we must have \(g_y \circ f \neq h_y \circ f\) and hence \(g_y(f(x)) \neq h_y(f(x))\), for some \(x \in X\). Since \(g_y\) and \(h_y\) only take different values at \(y\), it follows that \(f(x) = y\). Therefore \(f\) is surjective.

(b) Since the opposite category \(\text{Pop}\) of a pre-ordered set \(P\) is again a pre-ordered set, we can re-use the answer to question (4f): all the morphisms of \(P\) are epimorphisms.

(c) In the pre-ordered set \(Q\) from question 1(b), the unique morphism \(0 \to 1\) is both a monomorphism (by 4(f)) and an epimorphism (by 5(b)), but not an isomorphism, because there is no morphism from 1 to 0.

**Question 6**

(a) \((1, 0, \text{id}_1)\) is a terminal object, where \(1 = \{0\}\).

(b) Consider the object \((\mathbb{N}, 0, \text{succ})\) where \(\text{succ} \in \text{Set}(\mathbb{N}, \mathbb{N})\) is the successor function, \(\text{succ} n = n + 1\). This is initial in \(\mathbf{C}\), because for any object \((X, x_0, x_s)\), the function \(f : \mathbb{N} \to X\) recursively defined by

\[
\begin{align*}
f(0) &= x_0 \\
f(n+1) &= x_s(f(n))
\end{align*}
\]

gives a morphism \(f \in \mathbf{C}((\mathbb{N}, 0, \text{succ}), (X, x_0, x_s))\). It is the only such morphism, because if \(g \in \mathbf{C}((\mathbb{N}, 0, \text{succ}), (X, x_0, x_s))\), then \(g(0) = x_0\) and for all \(n \in \mathbb{N}\), \(g(n+1) = (g \circ \text{succ}) n = (x_s \circ g) n = x_s(g(n))\); hence by induction on \(n\), we have \((\forall n \in \mathbb{N}) g n = f n\).

**Question 7**

(a) Each element \(x \in X\) of a set \(X \in \text{Set}\) determines a point \(x^\triangledown : 1 \to X\) in \(\text{Set}\), namely the function mapping the unique element of 1 = \(\{0\}\) to \(x\). The mapping \(x \mapsto x^\triangledown\) is injective, since \(x^\triangledown 0 = x\); furthermore for every \(f \in \text{Set}(X, Y)\), \(f \circ x^\triangledown = \triangledown f x^\triangledown\). So if \((\forall p \in \text{Set}(1, X)) f \circ p = g \circ p\), then \((\forall x \in X) f x = g x\), that is, \(f = g\).

(b) \(\text{Set}^{\text{op}}\) is not well-pointed. Note that the empty set \(\emptyset\) is a terminal object in \(\text{Set}^{\text{op}}\) (because it is initial in \(\text{Set}\)) and that \(\text{Set}^{\text{op}}(\emptyset, X) = \text{Set}(X, \emptyset)\) is empty when \(X \neq \emptyset\). Then for example \(\text{id}_\mathbb{N} \neq \text{succ} \in \text{Set}^{\text{op}}(\mathbb{N}, \mathbb{N})\), but \((\forall p \in \text{Set}^{\text{op}}(\emptyset, \mathbb{N})) \text{id}_\mathbb{N} \circ p = \text{succ} \circ p\) is vacuously true.