Hoare Logic and Model Checking

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Course overview

This course is about **formal** techniques for validating software.

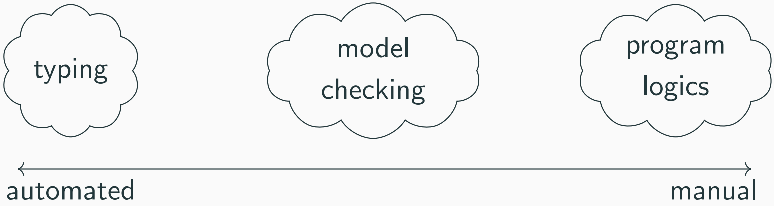
Formal methods allow us to **formally specify** the intended behaviour of our programs and use mathematical proof systems to **formally prove** that our programs satisfy their specification.

In this course we will focus on two techniques:

- **Hoare logic** (Lectures 1-6)
- **Model checking** (Lectures 7-12)

Course overview

There are many different formal reasoning techniques of varying expressivity and level of automation.



Formal vs. informal methods

Testing can quickly find obvious bugs:

- only trivial programs can be tested exhaustively
- the cases you do not test can still hide bugs
- coverage tools can help

Formal methods can improve assurance:

- allows us to reason about all possible executions
- can reveal hard-to-find bugs

Famous software bugs

At least 3 people were killed due to massive radiation overdoses delivered by a Therac-25 radiation therapy machine.

- the cause was a race-condition in the control software

An unmanned Ariane 5 rocket blew up on its maiden flight; the rocket and its cargo were estimated to be worth $500M.

- the cause was an unsafe floating point to integer conversion

Formal vs. informal methods

However, formal methods are not a panacea:

- formally verified designs may still not work
- can give a false sense of security
- formal verification can be very expensive and time-consuming

Formal methods should be used in conjunction with testing, not as a replacement.

Lecture plan

Lecture 1: Informal introduction to Hoare logic

Lecture 2: Formal semantics of Hoare logic

Lecture 3: Examples, loop invariants & total correctness

Lecture 4: Mechanised program verification

Lecture 5: Separation logic

Lecture 6: Examples in separation logic

Hoare logic

Hoare logic

Hoare logic is a formalism for relating the **initial** and **terminal** state of a program.

Hoare logic was invented in 1969 by Tony Hoare, inspired by earlier work of Robert Floyd.

Hoare logic is still an active area of research.

Hoare logic

Hoare logic uses **partial correctness triples** for specifying and reasoning about the behaviour of programs:

$$\{P\}\ C\ \{Q\}$$

Here C is a command and P and Q are state predicates.

- P is called the precondition and describes the initial state
- Q is called the postcondition and describes the terminal state

Hoare logic

To define a Hoare logic we need three main components:

- the programming language that we want to reason about, along with its operational semantics
- an assertion language for defining state predicates, along with a semantics
- a formal interpretation of Hoare triples, together with a (sound) formal proof system for deriving Hoare triples

This lecture will introduce each component informally.
In the coming lectures we will cover the formal details.

The WHILE language

The WHILE language

WHILE is a prototypical imperative language. Programs consists of commands, which include branching, iteration and assignments:

$$\begin{array}{rcl} C & ::= & \mathbf{skip} \mid C_1; C_2 \mid V := E \\ & \mid & \mathbf{if}\ B\ \mathbf{then}\ C_1\ \mathbf{else}\ C_2 \mid \mathbf{while}\ B\ \mathbf{do}\ C \end{array}$$

Here E is an expression which evaluates to a natural number and B is a boolean expression, which evaluates to a boolean.

States are mappings from variables to natural numbers.

The WHILE language

The grammar for expressions and boolean includes the usual arithmetic operations and comparison operators:

$$
\begin{array}{rcll}
E & ::= & N \mid V \mid E_1 + E_2 \mid & \textit{expressions} \\
  & \mid & E_1 - E_2 \mid E_1 \times E_2 \mid \cdots & \\
  & & & \\
B & ::= & T \mid F \mid E_1 = E_2 & \textit{boolean expressions} \\
  & \mid & E_1 \leq E_2 \mid E_1 \geq E_2 \mid \cdots &
\end{array}
$$

Note that expressions do not have side effects.

The assertion language

Hoare logic

State predicates P and Q can refer to program variables from C and will be written using standard mathematical notations together with **logical operators** like:

- $\wedge$ ("and"), $\vee$ ("or"), $\neg$ ("not") and $\Rightarrow$ ("implies")

For instance, the predicate $X = Y + 1 \wedge Y > 0$ describes states in which the variable Y contains a positive value and the value of X is equal to the value of Y plus 1.

Partial correctness triples

The partial correctness triple $\{P\}\ C\ \{Q\}$ holds if and only if:

- whenever C is executed in an initial state satisfying P
- and this execution terminates
- then the terminal state of the execution satisfies Q.

For instance,

- $\{X = 1\}\ X := X + 1\ \{X = 2\}$ holds
- $\{X = 1\}\ X := X + 1\ \{X = 3\}$ does not hold

Partial correctness

Partial correctness triples are called **partial** because they only specify the intended behaviour of terminating executions.

For instance, $\{X = 1\}$ **while** $X > 0$ **do** $X := X + 1$ $\{X = 0\}$ holds, because the given program never terminates when executed from an initial state where X is 1.

Hoare logic also features total correctness triples that strengthen the specification to require termination.

Total correctness

The total correctness triple $[P]\ C\ [Q]$ holds if and only if:

- whenever C is executed in an initial state satisfying P
- then the execution must terminate
- and the terminal state must satisfy Q.

There is no standard notation for total correctness triples, but we will use $[P]\ C\ [Q]$.

Total correctness

The following total correctness triple does not hold:

$$[X = 1]\ \textbf{while}\ X > 0\ \textbf{do}\ X := X + 1\ [X = 0]$$

- the loop never terminates when executed from an initial state where X is positive

The following total correctness triple does hold:

$$[X = 0]\ \textbf{while}\ X > 0\ \textbf{do}\ X := X + 1\ [X = 0]$$

- the loop always terminates immediately when executed from an initial state where X is zero

Total correctness

Informally: total correctness = termination + partial correctness.

It is often easier to show partial correctness and termination separately.

Termination is usually straightforward to show, but there are examples where it is not: no one knows whether the program below terminates for all values of X

$$\begin{array}{l} \textbf{while } X > 1 \textbf{ do} \\ \quad \textbf{if } \mathit{ODD}(X) \textbf{ then } X := 3 * X + 1 \textbf{ else } X := X \ \mathit{DIV} \ 2 \end{array}$$

Microsoft's T2 tool proves systems code terminates.

Specifications

Simple examples

$\{\perp\}\ C\ \{Q\}$

- this says nothing about the behaviour of C, because $\perp$ never holds for any initial state

$\{\top\}\ C\ \{Q\}$

- this says that whenever C halts, Q holds

$\{P\}\ C\ \{T\}$

- this holds for every precondition P and command C, because T always holds in the terminate state

Simple examples

$[P]\ C\ [T]$

- this says that C always terminates when executed from an initial state satisfying P

$[T]\ C\ [Q]$

- this says that C always terminates in a state where Q holds

Auxiliary variables

Consider a program C that computes the maximum value of two variables X and Y and stores the result in a variable Z.

Is this a good specification for C?

$$\{\top\}\ C\ \{(X \leq Y \Rightarrow Z = Y) \wedge (Y \leq X \Rightarrow Z = X)\}$$

No! Take C to be $X := 0; Y := 0; Z := 0$, then C satisfies the above specification. The postcondition should refer to the **initial** values of X and Y.

In Hoare logic we use **auxiliary variables** which do not occur in the program to refer to the initial value of variables in postconditions.

Auxiliary variables

For instance, $\{X = x \wedge Y = y\}\ C\ \{X = y \wedge Y = x\}$, expresses that if C terminates then it exchanges the values of variables X and Y.

Here x and y are auxiliary variables (or ghost variables) which are not allowed to occur in C and are only used to name the initial values of X and Y.

Informal convention: program variables are uppercase and auxiliary variables are lowercase.

Formal proof system for Hoare logic

Hoare logic

We will now introduce a natural deduction proof system for partial correctness triples due to Tony Hoare.

The logic consists of a set of **axiom schemas** and **inference rule schemas** for deriving consequences from premises.

If S is a statement of Hoare logic, we will write $\vdash S$ to mean that the statement S is derivable.

Hoare logic

The inference rules of Hoare logic will be specified as follows:

$$\frac{\vdash S_1 \qquad \cdots \qquad \vdash S_n}{\vdash S}$$

This expresses that S may be deduced from assumptions S_1, ..., S_n.

An axiom is an inference rule without any assumptions:

$$\frac{}{\vdash S}$$

In general these are schemas that may contain meta-variables.

Hoare logic

A proof tree for $\vdash S$ in Hoare logic is a tree with $\vdash S$ at the root, constructed using the inference rules of Hoare logic with axioms at the leaves.

$$\dfrac{\dfrac{\dfrac{}{\vdash S_1} \qquad \dfrac{}{\vdash S_2}}{\vdash S_3} \qquad \dfrac{}{\vdash S_4}}{\vdash S}$$

We typically write proof trees with the root at the bottom.

Formal proof system

$$\frac{}{\vdash \{P\}\ \textbf{skip}\ \{P\}} \qquad \frac{}{\vdash \{P[E/V]\}\ V := E\ \{P\}}$$

$$\frac{\vdash \{P\}\ C_1\ \{Q\} \qquad \vdash \{Q\}\ C_2\ \{R\}}{\vdash \{P\}\ C_1; C_2\ \{R\}}$$

$$\frac{\vdash \{P \wedge B\}\ C_1\ \{Q\} \qquad \vdash \{P \wedge \neg B\}\ C_2\ \{Q\}}{\vdash \{P\}\ \textbf{if}\ B\ \textbf{then}\ C_1\ \textbf{else}\ C_2\ \{Q\}}$$

$$\frac{\vdash \{P \wedge B\}\ C\ \{P\}}{\vdash \{P\}\ \textbf{while}\ B\ \textbf{do}\ C\ \{P \wedge \neg B\}}$$

Formal proof system

$$\frac{\vdash P_1 \Rightarrow P_2 \qquad \vdash \{P_2\}\ C\ \{Q_2\} \qquad \vdash Q_2 \Rightarrow Q_1}{\vdash \{P_1\}\ C\ \{Q_1\}}$$

$$\frac{\vdash \{P_1\}\ C\ \{Q\} \qquad \vdash \{P_2\}\ C\ \{Q\}}{\vdash \{P_1 \vee P_2\}\ C\ \{Q\}}$$

$$\frac{\vdash \{P\}\ C\ \{Q_1\} \qquad \vdash \{P\}\ C\ \{Q_2\}}{\vdash \{P\}\ C\ \{Q_1 \wedge Q_2\}}$$

The skip rule

$$\frac{}{\vdash \{P\}\ \textbf{skip}\ \{P\}}$$

The **skip** axiom expresses that any assertion that holds before **skip** is executed also holds afterwards.

P is a meta-variable ranging over an arbitrary state predicate.

For instance, $\vdash \{X = 1\}\ \textbf{skip}\ \{X = 1\}$.

The assignment rule

$$\overline{\vdash \{P[E/V]\}\ V := E\ \{P\}}$$

Here $P[E/V]$ means the assertion P with the expression E substituted for all occurences of the variable V.

For instance,

$$\{X + 1 = 2\}\ X := X + 1\ \{X = 2\}$$

$$\{Y + X = Y + 10\}\ X := Y + X\ \{X = Y + 10\}$$

The assignment rule

This assignment axiom looks backwards! Why is it sound?

In the next lecture we will prove it sound, but for now, consider some plausible alternative assignment axioms:

$$\frac{}{\vdash \{P\}\ V := E\ \{P[E/V]\}}$$

We can instantiate this axiom to obtain the following triple which does not hold:

$$\{X = 0\}\ X := 1\ \{1 = 0\}$$

The rule of consequence

$$\frac{\vdash P_1 \Rightarrow P_2 \qquad \vdash \{P_2\}\ C\ \{Q_2\} \qquad \vdash Q_2 \Rightarrow Q_1}{\vdash \{P_1\}\ C\ \{Q_1\}}$$

The rule of consequence allows us to strengthen preconditions and weaken postconditions.

Note: the $\vdash P \Rightarrow Q$ hypotheses are a different kind of judgment.

For instance, from $\{X + 1 = 2\}\ X := X + 1\ \{X = 2\}$
we can deduce $\{X = 1\}\ X := X + 1\ \{X = 2\}$.

Sequential composition

$$\frac{\vdash \{P\}\ C_1\ \{Q\} \qquad \vdash \{Q\}\ C_2\ \{R\}}{\vdash \{P\}\ C_1; C_2\ \{R\}}$$

If the postcondition of C_1 matches the precondition of C_2, we can derive a specification for their sequential composition.

For example, if one has deduced:

- $\{X = 1\}\ X := X + 1\ \{X = 2\}$
- $\{X = 2\}\ X := X + 1\ \{X = 3\}$

we may deduce that $\{X = 1\}\ X := X + 1; X := X + 1\ \{X = 3\}$.

The conditional rule

$$\frac{\vdash \{P \wedge B\}\ C_1\ \{Q\} \qquad \vdash \{P \wedge \neg B\}\ C_2\ \{Q\}}{\vdash \{P\}\ \textbf{if}\ B\ \textbf{then}\ C_1\ \textbf{else}\ C_2\ \{Q\}}$$

For instance, to prove that

$$\vdash \{T\}\ \textbf{if}\ X \geq Y\ \textbf{then}\ Z := X\ \textbf{else}\ Z := Y\ \{Z = max(X, Y)\}$$

It suffices to prove that $\vdash \{T \wedge X \geq Y\}\ Z := X\ \{Z = max(X, Y)\}$ and $\vdash \{T \wedge \neg(X \geq Y)\}\ Z := Y\ \{Z = max(X, Y)\}$.

The loop rule

$$\frac{\vdash \{P \wedge B\}\ C\ \{P\}}{\vdash \{P\}\ \textbf{while}\ B\ \textbf{do}\ C\ \{P \wedge \neg B\}}$$

The loop rule says that

- if P is an invariant of the loop body when the loop condition succeeds, then P is an invariant for the whole loop
- and if the loop terminates, then the loop condition failed

We will return to be problem of finding loop invariants.

Conjunction and disjunction rule

$$\frac{\vdash \{P_1\}\ C\ \{Q\} \qquad \vdash \{P_2\}\ C\ \{Q\}}{\vdash \{P_1 \vee P_2\}\ C\ \{Q\}}$$

$$\frac{\vdash \{P\}\ C\ \{Q_1\} \qquad \vdash \{P\}\ C\ \{Q_2\}}{\vdash \{P\}\ C\ \{Q_1 \wedge Q_2\}}$$

These rules are useful for splitting up proofs.

Any proof with these rules could be done without using them

- i.e. they are theoretically redundant (proof omitted)
- however, useful in practice

Summary

Hoare Logic is a formalism for reasoning about the behaviour of programs by relating their initial and terminal state.

It uses an assertion logic based on first-order logic to reason about program states and extends this with Hoare triples to reason about the programs.

Suggested reading:

- C. A. R. Hoare. An axiomatic basis for computer programming. 1969.
- R. W. Floyd. Assigning meanings to programs. 1967.