Relational extensionality

\[ R = S : A \rightarrow B \]

iff

\[ \forall a \in A. \forall b \in B. aRb \iff aSb \]
Relational composition

\[ R \xrightarrow{A} B \xrightarrow{S} C \mapsto A \xrightarrow{S \circ R} C \]

\[(a, c) \in (S \circ R) \iff \exists b \in B . (b, c) \in S \land (a, b) \in R\]

\[ a \circ (S \circ R) c \iff \exists b \in B . b \circ c \in (S \circ R) a \circ b \]
Theorem 102  Relational composition is associative and has the identity relation as neutral element.

- **Associativity.**

  For all \( R : A \rightarrow B, S : B \rightarrow C, \) and \( T : C \rightarrow D, \)

  \[ (T \circ S) \circ R = T \circ (S \circ R) \]

- **Neutral element.**

  For all \( R : A \rightarrow B, \)

  \[ R \circ \text{id}_A = R = \text{id}_B \circ R \]

\[ \def\id\{ \text{id} \} \]

It is cumbersome to write \( T \circ S \circ R \)
Relations and matrices

Definition 103

1. For positive integers $m$ and $n$, an $(m \times n)$-matrix $M$ over a semiring $(S, 0, \oplus, 1, \otimes)$ is given by entries $M_{i,j} \in S$ for all $0 \leq i < m$ and $0 \leq j < n$.

$$M = \begin{bmatrix}
0 & \cdots & i & \cdots & m-1 \\
\vdots & & & & \vdots \\
0 & \cdots & M_{i,j} & \cdots & 0
\end{bmatrix}$$

Theorem 104  Matrix multiplication is associative and has the identity matrix as neutral element.
\[ I_{(k \times k)} - \text{matrix} \quad I_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

\[ M_{(m \times n)} - \text{matrix} \]

\[ L_{(l \times m)} - \text{matrix} \]

\[ (M \circ L)_{i,j} = \bigoplus_{k=0}^{m-1} M_{k,i,j} \circ L_{i,k} \]

Consider the semiring of Booleans \((\{0,1\}, +, \cdot, 0, 1)\)

\[ (M \circ L)_{i,j} = \bigvee_{k=0}^{m-1} M_{k,i,j} \land L_{i,k} \Rightarrow \exists k. M_{k,i,j} \land L_{i,k} \]

\[ \left\langle \right. \]
\((m \times n)\)-matrices over \(\text{Booleans}\).

\[ [m] \mapsto [n] \]

\([r] = \{0, 1, \ldots, k-1\} \]

M \sim \rightarrow \text{rel}(M)

\text{Def } (i, j) \in \text{rel}(M) \iff M_{i,j} = 1

\text{mat}(R) \sim \rightarrow R

\text{Def } \text{mat}(R)_{i,j} = \begin{cases} 0 & (i,j) \not\in R \\ 1 & (i,j) \in R \end{cases}
Exercise

\[ M \hspace{1cm} \overset{\text{rel}(M)}{\longmapsto} \hspace{1cm} \text{mat}(\text{rel}(M)) \]

\[ \text{mat}(R) \hspace{1cm} \overset{\text{rel}(\text{mat}(R))}{\longmapsto} \hspace{1cm} \text{rel}(\text{mat}(R)) \]

\[ M = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \quad \text{with} \quad M_{ij} = 1 \]

\[ \text{mat}(\text{rel}(M)) = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \]

\[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]
Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication.

\[ M : (m \times n) \text{-mat} \quad \longrightarrow \quad \text{rel}(M) : [m] \rightarrow [n] \]

\[ L : (l \times m) \text{-mat} \quad \longrightarrow \quad \text{rel}(L) : [l] \rightarrow [m] \]

\[ M \cdot L : (l \times n) \text{-mat} \quad \longrightarrow \quad \text{rel}(M \circ L) : [l] \rightarrow [n] \]
Directed graphs

Definition 108  A **directed graph** \((A, R)\) consists of a set \(A\) and a relation \(R\) on \(A\) (i.e. a relation from \(A\) to \(A\)).
Corollary 110 For every set $A$, the structure $(\mathcal{R}(A), \text{id}_A, \circ)$ is a monoid.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a.(R \circ R)^{n-1}b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\text{id}_A$</td>
</tr>
<tr>
<td>1</td>
<td>$a.Rb$</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>$a.(R \circ R)^{n-2}b$</td>
</tr>
</tbody>
</table>

Definition 111 For $R \in \mathcal{R}(A)$ and $n \in \mathbb{N}$, we let

$$R^{\circ n} = \underbrace{R \circ \cdots \circ R}_{n \text{ times}} \in \mathcal{R}(A)$$

be defined as $\text{id}_A$ for $n = 0$, and as $R \circ R^{\circ m}$ for $n = m + 1$. 
Proposition 113  Let \((A, R)\) be a directed graph. For all \(n \in \mathbb{N}\) and \(s, t \in A\), \(s R^n t\) iff there exists a path of length \(n\) in \(R\) with source \(s\) and target \(t\).

**Proof:**

By induction,

Base case \((n = 0)\):

\[ R^0 = \text{id} \]

\(s R^0 t \iff \exists \text{ path of length } 0 \text{ from } s \text{ to } t \)

\(s = t \)
Inductive step: \( n = m + 1 \)

**Inductive Hypothesis (IH):** Assume \( s R^m t \iff \exists \text{ paths of length } m \text{ from } s \text{ to } t \)

\( \therefore s R^{m+1} t \iff \exists \text{ paths of length } m+1 \text{ from } s \text{ to } t \)

\[ s (R_0 R^m) t \]

\[ \exists p \cdot (p R^m t) \wedge (s R p) \iff \exists \text{ paths of length } m \text{ from } p \text{ to } t \wedge s R p \]
Definition 114  For $R \in \text{Rel}(A)$, let
\[ R^{o*} = \bigcup \{ R^{on} \in \text{Rel}(A) \mid n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} R^{on}. \]

Corollary 115  Let $(A, R)$ be a directed graph. For all $s, t \in A$, $s \ R^{o*} t$ iff there exists a path with source $s$ and target $t$ in $R$. 