Given a set $\mathcal{U}$, we are considering the powerset $\mathcal{P}(\mathcal{U})$.

it is a Boolean algebra $\emptyset, \mathcal{U}, \cup, \cap$

$\cup$ union | $\cap$ intersection

$\mathcal{F} \subseteq \mathcal{P}\mathcal{U}$

big unions $\quad \bigcup \mathcal{F} \in \mathcal{P}\mathcal{U}$

big intersection $\quad \bigcap \mathcal{F} \in \mathcal{P}\mathcal{U}$

<u>NB</u>: If $\mathcal{F} = \{A, B\}$ then $\bigcup \mathcal{F} = A \cup B$

$\bigcap \mathcal{F} = A \cap B$

If $\mathcal{F} = \{\ \}$ then $\bigcup \mathcal{F} = \emptyset$, $\underline{\text{Exercise}}$ $\bigcap \mathcal{F} = \mathcal{U}$

Big intersections

Definition 92 *Let $\mathcal{U}$ be a set. For a collection of sets $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$, we let the* big intersection *(relative to $\mathcal{U}$) be defined as*

$$\bigcap \mathcal{F} = \{ x \in \mathcal{U} \mid \forall A \in \mathcal{F}. x \in A \} \quad .$$

Theorem 93 *Let*

$$\mathcal{F} = \Big\{ S \subseteq \mathbb{R} \;\Big|\; (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \Big\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

PROOF: because $\mathbb{N} \subseteq \mathbb{R}$

and $0 \in \mathbb{N}$

$\forall x \in \mathbb{R}.\ x \in \mathbb{N} \Rightarrow x+1 \in \mathbb{N}$

E.g. $\mathbb{Z} \in \mathcal{F}$

$\mathbb{Q} \in \mathcal{F}$

Rem: $\bigcap \mathcal{F} \subseteq \mathbb{N}$

$\bigcap \mathcal{F} \subseteq \mathbb{Z}$

$\bigcap \mathcal{F} \subseteq \mathbb{Q}$

Rem: $\forall A \in \mathcal{F}.\ \bigcap \mathcal{F} \subseteq A$

$\mathbb{N} \subseteq \bigcap \mathcal{F}$

$\Leftrightarrow \forall n \in \mathbb{N}.\ n \in \bigcap \mathcal{F}$

Show it by induction on $\mathbb{N}$.

Remark For any two sets A, B we can consider $\mathcal{F} = \{A, B\}$ and its big union $\bigcup\{A, B\}$ which we write as $A \cup B$ and is defined by

$$x \in A \cup B \iff (x \in A) \vee (x \in B)$$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}. x \in X$$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \left(\forall X \in \mathcal{F}. x \in X\right) \quad .$$

Cf in ML

(α, β) disunion = <u>one</u> of α | <u>two</u> of β ;

Disjoint unions

Definition 94 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \;=\; \big(\{1\} \times A\big) \cup \big(\{2\} \times B\big) \quad .$$

Thus,

$$\forall x.\, x \in (A \uplus B) \iff \big(\,\exists a \in A.\, x = (1, a)\,\big) \vee \big(\,\exists b \in B.\, x = (2, b)\,\big)\,.$$

Proposition 96 *For all finite sets* A *and* B,

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \enspace .$$

PROOF IDEA:

$A = \{a_1, \ldots, a_n\}$ $\#A = n$ $\quad \forall i, j \; a_i \neq b_j$

$B = \{b_1, \ldots, b_m\}$ $\#B = m$

$A \cup B = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$

$\#(A \cup B) = n + m.$

Corollary 97 *For all finite sets* A *and* B,

$$\#(A \uplus B) = \#A + \#B \enspace .$$

Relations

Definition 99 *A* (binary) relation R *from a set* A *to a set* B

$$R : A \longrightarrow\!\!\!\!\!\!\!+\;\; B \qquad \textit{or} \qquad R \in \mathrm{Rel}(A, B) \quad ,$$

is

$$R \subseteq A \times B \qquad \textit{or} \qquad R \in \mathcal{P}(A \times B) \quad .$$

Notation 100 *One typically writes* $a\,R\,b$ *for* $(a, b) \in R$.

Informal examples:

- Computation.
- Typing.
- Program equivalence.
- Networks.
- Databases.

Examples: $1 \, R_2 \, 1$, $1 \, R_2 \, {-1}$, ...

$(1,1) \in R_2$ $(1,-1) \in R_2$

Examples:

- Empty relation.

  $\emptyset : A \twoheadrightarrow B$ $\qquad (a \; \emptyset \; b \iff \textbf{false})$

- Full relation.

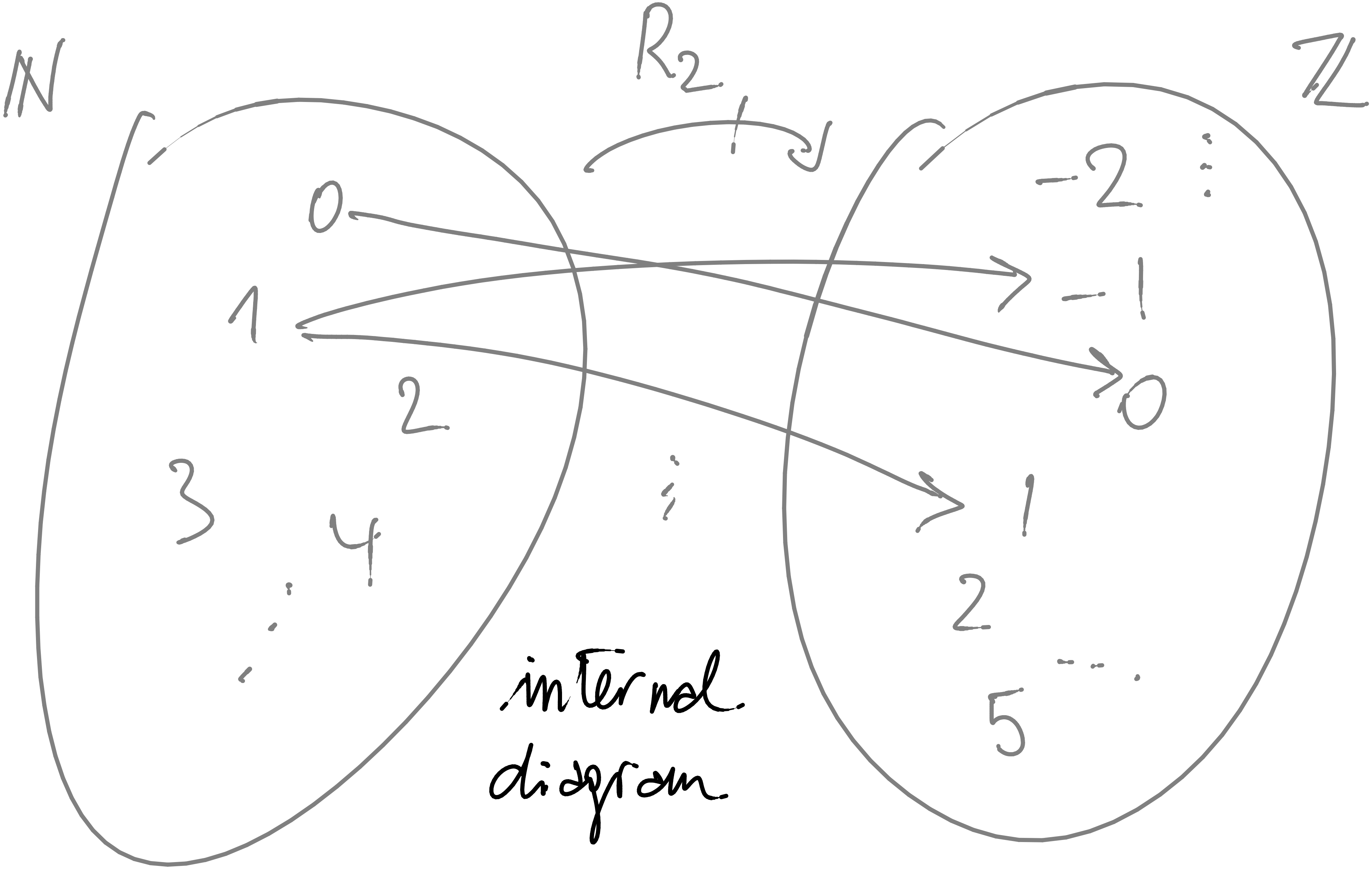
  $(A \times B) : A \twoheadrightarrow B$ $\qquad (a \; (A \times B) \; b \iff \textbf{true})$

- Identity (or equality) relation.

  $\mathrm{id}_A = \{ (a, a) \mid a \in A \} : A \twoheadrightarrow A$ $\qquad (a \; \mathrm{id}_A \; a' \iff a = a')$

- Integer square root.

  $R_2 = \{ (m, n) \mid m = n^2 \} : \mathbb{N} \twoheadrightarrow \mathbb{Z}$ $\qquad (m \; R_2 \; n \iff m = n^2)$



$(0,0) \in R_2$, $(1,1) \in R_2$, $(1,-1) \in R_2$

Rem: $R : A \nrightarrow B$

$R^{op} : B \nrightarrow A$

defined by

$\forall\ b \in B \quad a \in A$

$$b \, R^{op} \, a \overset{def}{\iff} a \, R \, b$$

Internal diagrams

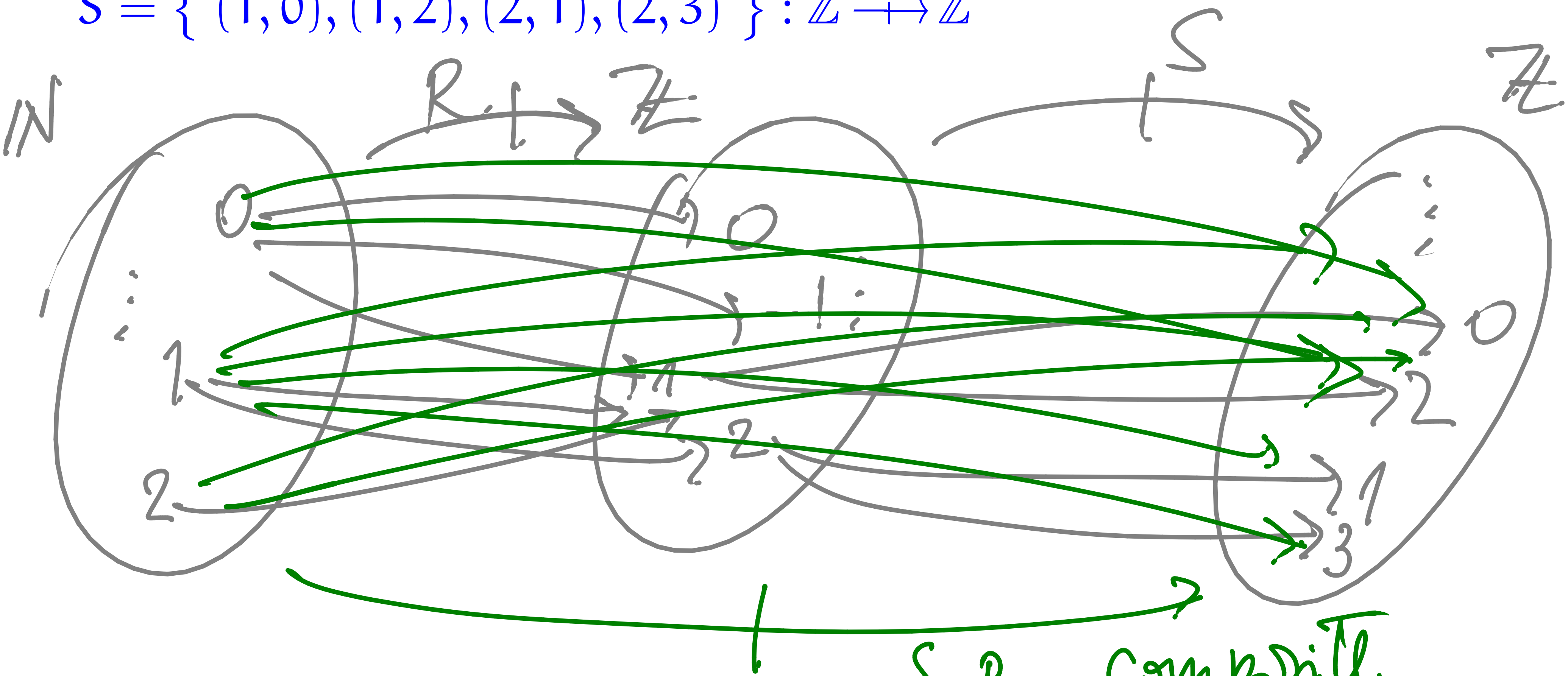
$$A \xrightarrow[\mapstochar]{R} B \xrightarrow[\mapstochar]{S} C$$

$$A \xrightarrow[\mapstochar]{S \circ R} C$$

Example:

$$R = \{\ (0,0), (0,-1), (0,1), (1,2), (1,1), (2,1)\ \} : \mathbb{N} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{Z}$$

$$S = \{\ (1,0), (1,2), (2,1), (2,3)\ \} : \mathbb{Z} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{Z}$$



$S \circ R$ composite