We can collect all subsets of a given set, say \( U \), into a new set which is called the powerset and denoted \( \mathcal{P}(U) \).

**Powerset axiom**

For any set, there is a set consisting of all its subsets.

\[
\forall X. \ X \in \mathcal{P}(U) \iff X \subseteq U.
\]

Recall

\[
A \subseteq B \iff (\forall x. \ x \in A \Rightarrow x \in B) \iff \forall x \in A. \ x \in B
\]
The powerset construction increases cardinality

\[ \# \emptyset = 0 \]
\[ \# P(\emptyset) = 2^0 = 1 \]
\[ \# P P \emptyset = 2^1 = 2 \]
\[ \vdots \]
\[ \# P P \ldots P (\emptyset) = 2^n \]

Recall
\[ \# A = a \]
\[ \Rightarrow \# P A = 2^a \]
Venn diagrams

Union

Intersection

Complement
The powerset Boolean algebra

\[(\mathcal{P}(U), \emptyset, U, \cup, \cap, (\cdot)^c)\]

For all \(A, B \in \mathcal{P}(U)\),

\[A \cup B = \{x \in U \mid x \in A \lor x \in B\} \in \mathcal{P}(U)\]

\[A \cap B = \{x \in U \mid x \in A \land x \in B\} \in \mathcal{P}(U)\]

\[A^c = \{x \in U \mid \neg(x \in A)\} \in \mathcal{P}(U)\]

\[\text{notation: } x \notin A\]
## Sets and logic

<table>
<thead>
<tr>
<th>$\mathcal{P}(\mathbb{U})$</th>
<th>{false, true}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>false</td>
</tr>
<tr>
<td>$\mathbb{U}$</td>
<td>true</td>
</tr>
<tr>
<td>$\cup$</td>
<td>$\lor$</td>
</tr>
<tr>
<td>$\cap$</td>
<td>$\land$</td>
</tr>
<tr>
<td>$(\cdot)^c$</td>
<td>$\neg(\cdot)$</td>
</tr>
</tbody>
</table>
Proposition 85  Let \( U \) be a set and let \( A, B \in \mathcal{P}(U) \).

1. \( \forall X \in \mathcal{P}(U). \ A \cup B \subseteq X \iff (A \subseteq X \land B \subseteq X) \).

2. \( \forall X \in \mathcal{P}(U). \ X \subseteq A \cap B \iff (X \subseteq A \land X \subseteq B) \).

**Proof:** Let \( x \in \mathcal{P}(U) \); that is, \( x \subseteq U \).

\((\implies)\) Assume \( A \cup B \subseteq x \).

\( A \subseteq A \cup B \subseteq x \implies A \subseteq x \) by Lemma: \( \forall A. \ A \subseteq A \cup B \).

\( A \subseteq x \land B \subseteq x \implies A \cap B \subseteq x \) by Lemma: \( \forall A. \ B \subseteq A \implies B \subseteq A \).

\((\iff)\) Assume \( A \subseteq x \) and \( B \subseteq x \).

Proving only one \( A \cup B \subseteq x \).

\((\iff)\) Assume \( A \subseteq x \) and \( B \subseteq x \)
If $x \in A \cup B$, then $x \in X$.

This is, $\forall x. (x \in A) \lor (x \in B) \implies x \in X$.

Assume $x \in U$ such that $(x \in A) \lor (x \in B)$.

(1) If $x \in A$ and so, since $A \subseteq X$, $x \in X$.

(2) If $x \in B$ then $x \in X$ because $B \subseteq X$. 

\( \square \)
Corollary 86 Let $U$ be a set and let $A, B, C \in \mathcal{P}(U)$.

1. $C = A \cup B$

   iff

   $[A \subseteq C \land B \subseteq C]$
   $\land$

   $[\forall X \in \mathcal{P}(U). (A \subseteq X \land B \subseteq X) \implies C \subseteq X]$

2. $C = A \cap B$

   iff

   $[C \subseteq A \land C \subseteq B]$
   $\land$

   $[\forall X \in \mathcal{P}(U). (X \subseteq A \land X \subseteq B) \implies X \subseteq C]$

Proof principle for showing a set is a union or an intersection.
The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) , \quad A \cup B = B \cup A , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) , \quad A \cap B = B \cap A , \quad A \cap A = A$$
The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \ A \cup B = B \cup A \ , \ A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \ A \cap B = B \cap A \ , \ A \cap A = A$$

The empty set $\emptyset$ is a neutral element for $\cup$ and the universal set $U$ is a neutral element for $\cap$.

$$\emptyset \cup A = A = U \cap A$$
The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $U$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$U \cup A = U$$

With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) = A = A \cap (A \cup B)$$
The complement operation $(\cdot)^c$ satisfies complementation laws.

\[ A \cup A^c = U, \quad A \cap A^c = \emptyset \]

Exercise: Prove the De Morgan laws:

\[ (A \cup B)^c = A^c \cap B^c \]
\[ (A \cap B)^c = A^c \cup B^c \]
For every $a$ and $b$, there is a set with $a$ and $b$ as its only elements.

\[ \{ a, b \} \]

defined by

\[ \forall x. x \in \{ a, b \} \iff (x = a \lor x = b) \]

NB The set $\{ a, a \}$ is abbreviated as $\{ a \}$, and referred to as a singleton.
Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$
Exercise \( \langle a, b \rangle = \langle x, y \rangle \iff a = x \land b = y \)

(Cor: \( \langle a, b \rangle = \langle b, a \rangle \iff a = b \))

**Ordered pairing**

For every pair \( a \) and \( b \), the set

\[
\{ \{ a \}, \{ a, b \} \}
\]

is abbreviated as

\( \langle a, b \rangle \)

and referred to as an **ordered pair**.

E.g. \( \langle 1, 2 \rangle = \{ \{ 1 \}, \{ 1, 2 \} \} \neq \langle 2, 1 \rangle = \{ \{ 2 \}, \{ 2, 1 \} \} \)

\( \langle b, a \rangle \) not necessarily the same.