Extensionality axiom

Two sets are equal if they have the same elements.

Thus,

$$\forall \text{ sets } A, B.\ A = B \iff (\forall x.\, x \in A \iff x \in B) \ .$$

Example:

$$\{0\} \neq \{0, 1\} = \{1, 0\} \neq \{2\} = \{2, 2\}$$

Subsets and supersets

$$A = B \text{ iff } \boxed{(\forall x.\ x \in A \Rightarrow x \in B)}$$
$$\wedge\ (\forall x.\ x \in B \Rightarrow x \in A).$$

$$A \subseteq B \ \stackrel{\text{def}}{=}\ (\forall x.\ x \in A \Rightarrow x \in B)$$

$$A = B \iff (A \subseteq B) \wedge (B \subseteq A)$$

Lemma 83

Intuition $X \subseteq Y$

1. *Reflexivity.*

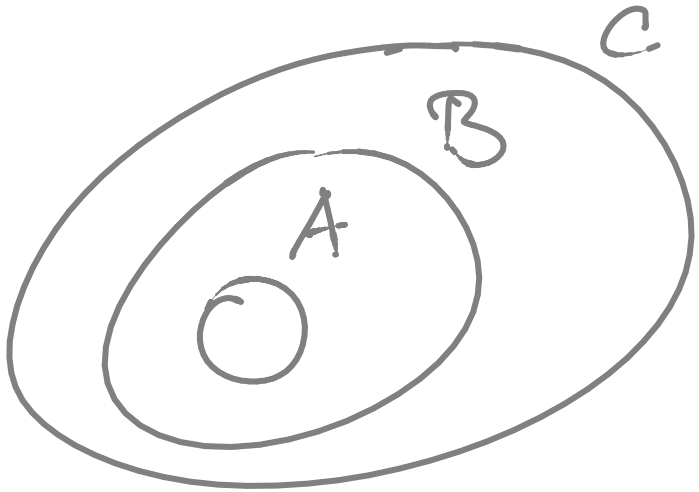
   For all sets A, $A \subseteq A$.

2. *Transitivity.*

   For all sets A, B, C, $(A \subseteq B \wedge B \subseteq C) \implies A \subseteq C$.

3. *Antisymmetry.*

   For all sets A, B, $(A \subseteq B \wedge B \subseteq A) \implies A = B$.



$$(A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$$

Assume ① $\boxed{A \subseteq B}$ and ② $\boxed{B \subseteq C}$

RTP: $A \subseteq C \overset{\text{def}}{\Longleftrightarrow} \forall x.\ x \in A \Rightarrow x \in C$

Let x be arbitrary

RTP: $x \in A \Rightarrow x \in C$

Assume: ③ $\boxed{x \in A}$.

RTP: $x \in C$

From ① and ③, $\boxed{x \in B}$ ④

From ④ and ②, $x \in C$

as required. $\boxtimes$

$a \in \{x \in A \mid P(x)\}$

$\overset{\text{def}}{\iff} (a \in A) \wedge P(a)$

NB: $\{x \in A \mid P(x)\} \subseteq A$

Separation principle

For any set A and any definable property P, there is a set containing precisely those elements of A for which the property P holds.

$$\{x \in A \mid P(x)\}$$

Example: $\mathbb{N}$ the set of natural numbers.

$\mathbb{Z}$ the set of integers

$\mathbb{N}_{odd} = \{ n \in \mathbb{N} \mid \exists k \in \mathbb{N}.\ n = 2k+1 \}$

$[-2..5] = \{ n \in \mathbb{Z} \mid -2 \leq n \leq 5 \}$

<u>Notation</u> by pattern matching

$\{ 2k+1 \in \mathbb{N} \mid k \in \mathbb{N} \}$

eg.

$\{ 2\ell \mid \ell \in \mathbb{N} \} \rightsquigarrow \{ n \in \mathbb{N} \mid \exists \ell \in \mathbb{N}.\ n = 2\ell \}$

notation for.

Russell's paradox

$\boxed{?}$ Does $R := \{x \mid \underbrace{x \notin x}_{P(x)}\}$ make sense as a set?

Let R be a set, and consider whether $R \in R$?

But
$$R \in R \iff P(R) \overset{\text{def}}{\iff} R \notin R$$

Hence it does not make sense to consider R as a set. $\boxtimes$

$$a \in \{x \mid P(x)\} \iff P(a).$$

E.g.: $\{x \in A \mid \text{false}\}$

$a \in \{x \in A \mid \text{false}\} \iff \text{false}$

Empty set

$$\emptyset \quad \text{or} \quad \{\,\}$$

For all sets A,

$A \subseteq A$

and

$\emptyset \subseteq A$

defined by

$$\forall\, x.\, x \notin \emptyset$$

or, equivalently, by

$$\neg(\exists\, x.\, x \in \emptyset)$$

Eg: $\#\{n \in \mathbb{N} \mid 0 \leq n \leq 7\} = 8$.

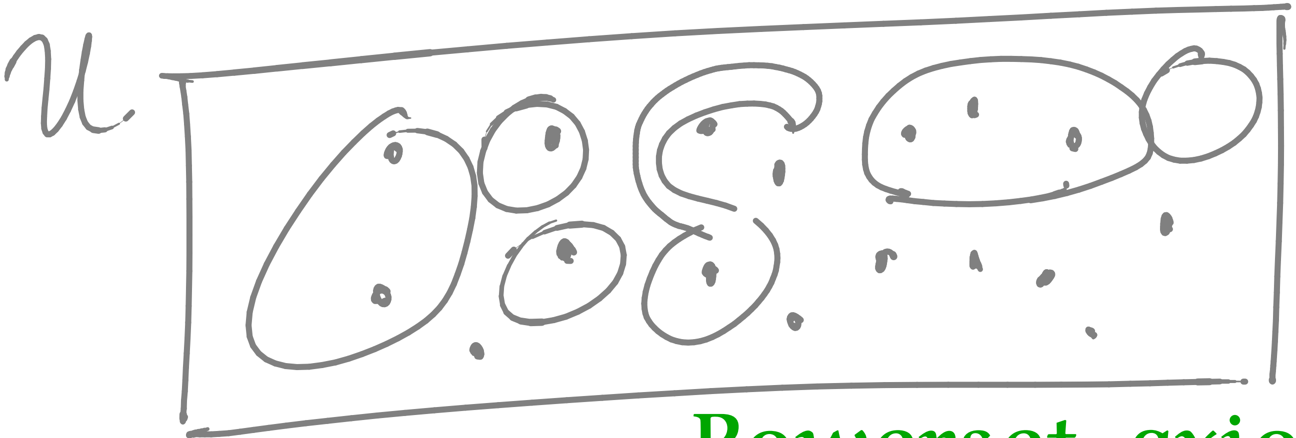
Cardinality

The *cardinality* of a set specifies its size. If this is a natural number, then the set is said to be *finite*.

Typical notations for the cardinality of a set S are $\#S$ or $|S|$.

Example:

$$\#\emptyset = 0$$



Powerset axiom

For any set, there is a set consisting of all its subsets.

$\mathcal{P}(U)$ — powerset of U

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U \quad .$$

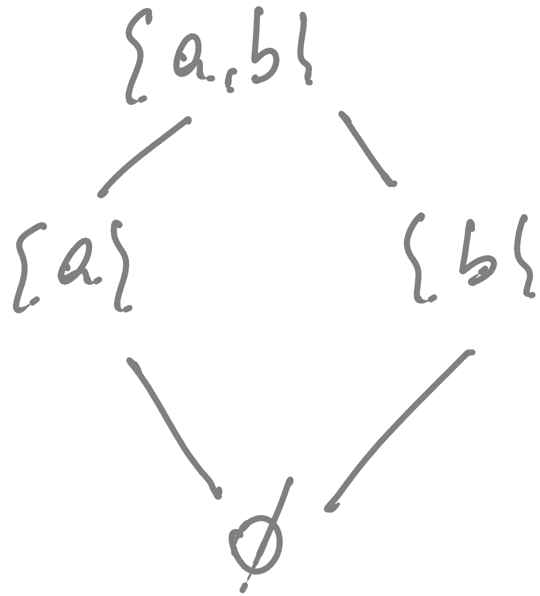
Hasse diagrams

Example: $\mathcal{U} = \{0\} \rightsquigarrow \mathcal{P}(\mathcal{U}) = \{\emptyset, \{0\}\}$

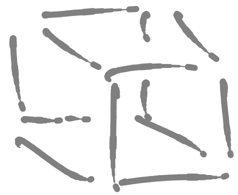
$\{0\}$

$\emptyset$

$\mathcal{U} = \{a, b\} \rightsquigarrow \mathcal{P}(\mathcal{U}) =$



$\mathcal{U} = \{x, y, z\} \rightsquigarrow$ $\mathcal{P}(\mathcal{U})$



$\# \mathcal{U} = n \Rightarrow \# \mathcal{P}(\mathcal{U}) = 2^n$.

Proposition 84 *For all finite sets* U*,*

$$\#\,\mathcal{P}(U) = 2^{\#U} \quad .$$

PROOF IDEA: $U = \{a_1, a_2, \ldots, a_n\}$ $(n \in \mathbb{N})$.

$$S \subseteq U \qquad \begin{array}{cccccc} 0 & 1 & 0 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & & \cdots & a_n \end{array} \longleftrightarrow \{a_2, a_4, \ldots, a_n\}$$

$$\emptyset \subseteq U \longleftrightarrow \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ a_1 & a_2 & \cdots & a_n \end{array}$$

$$\{a_i\} \subseteq U \longleftrightarrow \begin{array}{cccccc} 0 & 0 & \cdots & 1 & \cdots & 0 \\ a_1 & a_2 & \cdots & a_i & \cdots & a_n \end{array}$$

subsets of U $\Longleftrightarrow$ sequences of $0,1$'s of length n

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; 2^n$

$\boxtimes$