Existential quantification

Existential statements are of the form

**there exists** an individual $x$ in the universe of discourse for which the property $P(x)$ holds

or, in other words,

**for some** individual $x$ in the universe of discourse, the property $P(x)$ holds

or, in symbols,

$$\exists x. P(x)$$
Example: The Pigeonhole Principle.

Let $n$ be a positive integer. If $n + 1$ letters are put in $n$ pigeonholes then there will be a pigeonhole with more than one letter.
Theorem 21 (Intermediate value theorem)  Let \( f \) be a real-valued continuous function on an interval \([a, b]\). For every \( y \) in between \( f(a) \) and \( f(b) \), there exists \( v \) in between \( a \) and \( b \) such that \( f(v) = y \).

Intuition:
The main proof strategy for existential statements:

To prove a goal of the form

$$\exists x. \ P(x)$$

find a \textit{witness} for the existential statement; that is, a value of \(x\), say \(w\), for which you think \(P(x)\) will be true, and show that indeed \(P(w)\), i.e. the predicate \(P(x)\) instantiated with the value \(w\), holds.
Proof pattern:
In order to prove

$$\exists x. P(x)$$

1. Write: Let $w = \ldots$ (the witness you decided on).
2. Provide a proof of $P(w)$. 
Scratch work:

Before using the strategy

**Assumptions**

\[ \exists x. P(x) \]

:**Goal**

After using the strategy

**Assumptions**

\[ \vdots \]

**Goals**

\[ P(w) \]

\[ \vdots \]

\[ w = \ldots \text{(the witness you decided on)} \]
Proposition 22  For every positive integer \( k \), there exist natural numbers \( i \) and \( j \) such that \( 4 \cdot k = i^2 - j^2 \).

**Proof:**

**WPD:** For every positive integer \( k \).  
\[ 4k = i^2 - j^2. \]

Assume \( k \) is an arbitrary positive integer.  

**RTD:** Find \( w \) a nat, find \( v \) a nat.

**RTD:** \( 4k = w^2 - v^2 \).
The use of existential statements:

To use an assumption of the form $\exists x. P(x)$, introduce a new variable $x_0$ into the proof to stand for some individual for which the property $P(x)$ holds. This means that you can now assume $P(x_0)$ true.
Theorem 24  For all integers \( l, m, n \), if \( l \mid m \) and \( m \mid n \) then \( l \mid n \).

**Proof:**

\[ \forall \text{int. } l, m, n. \\
( l \mid m \land m \mid n ) \Rightarrow l \mid n \]

Equivalent:

\[ \forall \text{int. } l, m, n. \\
\left[ (\exists i. \ i l = m ) \land (\exists j. \ j m = n ) \right] \Rightarrow (\exists k. \ kl = n ) \]

Let \( l, m, n \) be int.

Assume \( (\exists i. \ i l = m ) \) and \( (\exists j. \ j m = n ) \)

\[ \Rightarrow \exists k. \ kl = n \]

\[ \Rightarrow \exists w. \ w \cdot l = n \]

By \( \circ \) consider \( u \) s.t. \( u \cdot l = m \)
By (2) consider \( v \) s.t. \( v : m = n \)

Note that \( u : l = m \) so \( u : l : v = v : m = n \)

Hence \( w = u : l \) satisfies \( w : l = n \). \( \square \)
Unique existence

The notation

\[ \exists! \, x. \, P(x) \]

stands for

the *unique existence* of an \( x \) for which the property \( P(x) \) holds.

That is,

\[ \exists x. \, P(x) \land (\forall y. \forall z. (P(y) \land P(z)) \Rightarrow y = z) \]
Disjunction

Disjunctive statements are of the form

\[ P \lor Q \]

or, in other words,

either \( P \), \( Q \), or both hold

or, in symbols,

\[ P \lor Q \]
The main proof strategy for disjunction:

To prove a goal of the form
\[ P \lor Q \]
you may

1. try to prove \( P \) (if you succeed, then you are done); or
2. try to prove \( Q \) (if you succeed, then you are done); otherwise
3. break your proof into cases; proving, in each case, either \( P \) or \( Q \).
Proposition 25  For all integers $n$, either $n^2 \equiv 0 \pmod{4}$ or $n^2 \equiv 1 \pmod{4}$.

**Proof:** \( \forall \text{int } n. \ (n^2 \equiv 0 \pmod{4}) \lor (n^2 \equiv 1 \pmod{4}) \)

Consider arbitrary $n$ an integer.

1. We try to show $n^2 \equiv 0 \pmod{4}$ — we cannot!
2. We try to show $n^2 \equiv 1 \pmod{4}$ — we cannot!
3. We break the proof in cases, simply to establish either disjunct.

We look at two cases:

(i) $n$ is even, that is of the form $2i$ for an int $i$.

So $n^2 = (2i)^2 = 4i^2$ and we are done.
\( n^2 \equiv 0 \pmod{4} \)

(ii) \( n \) is odd; That is, \( n = 2j + 1 \) for some \( j \).

So, \[ n^2 = (2j + 1)^2 = 4j^2 + 4j + 1 \]

\[ = 4(j^2 + j) + 1 \]

Hence, \( n^2 - 1 = 4(j^2 + j) \) and \( n^2 \equiv 1 \pmod{4} \).

That is, \( n^2 \equiv 1 \pmod{4} \). \( \Box \)
The use of disjunction:

To use a disjunctive assumption

\[ P_1 \lor P_2 \]

to establish a goal \( Q \), consider the following two cases in turn: (i) assume \( P_1 \) to establish \( Q \), and (ii) assume \( P_2 \) to establish \( Q \).
Scratch work:

Before using the strategy

\[
\begin{align*}
\text{Assumptions} & \quad \text{Goal} \\
& \quad Q \\
& \quad \vdots \\
& \quad P_1 \lor P_2
\end{align*}
\]

After using the strategy

\[
\begin{align*}
\text{Assumptions} & \quad \text{Goal} & \quad \text{Assumptions} & \quad \text{Goal} \\
& \quad Q & \quad & \quad Q \\
& \quad \vdots & \quad & \quad \vdots \\
& \quad P_1 & \quad & \quad P_2
\end{align*}
\]
Proof pattern:
In order to prove $Q$ from some assumptions amongst which there is

$$P_1 \lor P_2$$

write: We prove the following two cases in turn: (i) that assuming $P_1$, we have $Q$; and (ii) that assuming $P_2$, we have $Q$. Case (i): Assume $P_1$. and provide a proof of $Q$ from it and the other assumptions. Case (ii): Assume $P_2$. and provide a proof of $Q$ from it and the other assumptions.