Complexity Theory
Lecture 6

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Easter Term 2017

http://www.cl.cam.ac.uk/teaching/1617/Complexity/
Clique

Given a graph $G = (V, E)$, a subset $X \subseteq V$ of the vertices is called a clique, if for every $u, v \in X$, $(u, v)$ is an edge.

As with IND, we can define a decision problem:

**CLIQUE** is defined as:

The set of pairs $(G, K)$, where $G$ is a graph, and $K$ is an integer, such that $G$ contains a clique with $K$ or more vertices.
Clique 2

CLIQUE is in \textbf{NP} by the algorithm which \textit{guesses} a clique and then verifies it.

CLIQUE is \textbf{NP}-complete, since

\textbf{IND} \leq_p CLIQUE

by the reduction that maps the pair \((G, K)\) to \((\bar{G}, K)\), where \(\bar{G}\) is the complement graph of \(G\).
**$k$-Colourability**

A graph $G = (V, E)$ is $k$-colourable, if there is a function

$$\chi : V \to \{1, \ldots, k\}$$

such that, for each $u, v \in V$, if $(u, v) \in E$,

$$\chi(u) \neq \chi(v)$$

This gives rise to a decision problem for each $k$.

2-colourability is in $P$.

For all $k > 2$, $k$-colourability is $NP$-complete.
3-Colourability

3-Colourability is in NP, as we can guess a colouring and verify it.

To show NP-completeness, we can construct a reduction from 3SAT to 3-Colourability.

For each variable $x$, we have two vertices $x, \bar{x}$ which are connected in a triangle with the vertex $a$ (common to all variables).

In addition, for each clause containing the literals $l_1, l_2$ and $l_3$ we have a gadget.
With a further edge from $a$ to $b$. 
Hamiltonian Graphs

Recall the definition of $\text{HAM}$—the language of Hamiltonian graphs.

Given a graph $G = (V, E)$, a Hamiltonian cycle in $G$ is a path in the graph, starting and ending at the same node, such that every node in $V$ appears on the cycle $\textit{exactly once}$.

A graph is called Hamiltonian if it contains a Hamiltonian cycle.

The language $\text{HAM}$ is the set of encodings of Hamiltonian graphs.
Hamiltonian Cycle

We can construct a reduction from 3SAT to HAM

Essentially, this involves coding up a Boolean expression as a graph, so that every satisfying truth assignment to the expression corresponds to a Hamiltonian circuit of the graph.

This reduction is much more intricate than the one for IND.
Travelling Salesman

Recall the travelling salesman problem

Given

- \( V \) — a set of nodes.
- \( c : V \times V \to \mathbb{N} \) — a cost matrix.

Find an ordering \( v_1, \ldots, v_n \) of \( V \) for which the total cost:

\[
c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})
\]

is the smallest possible.
Travelling Salesman

As with other optimisation problems, we can make a decision problem version of the Travelling Salesman problem.

The problem $\text{TSP}$ consists of the set of triples

$$(V, c : V \times V \rightarrow \mathbb{N}, t)$$

such that there is a tour of the set of vertices $V$, which under the cost matrix $c$, has cost $t$ or less.
Reduction

There is a simple reduction from \textsc{HAM} to \textsc{TSP}, mapping a graph \((V, E)\) to the triple \((V, c : V \times V \to \mathbb{N}, n)\), where

\[
c(u, v) = \begin{cases} 
1 & \text{if } (u, v) \in E \\
2 & \text{otherwise}
\end{cases}
\]

and \(n\) is the size of \(V\).