

# Recall: $\lambda$ -Terms, $M$

are built up from a given, countable collection of

- ▶ variables  $x, y, z, \dots$

by two operations for forming  $\lambda$ -terms:

- ▶  $\lambda$ -abstraction:  $(\lambda x.M)$   
(where  $x$  is a variable and  $M$  is a  $\lambda$ -term)
- ▶ application:  $(M M')$   
(where  $M$  and  $M'$  are  $\lambda$ -terms).

Some random examples of  $\lambda$ -terms:

$x$     $(\lambda x.x)$     $((\lambda y.(x y))x)$     $(\lambda y.((\lambda y.(x y))x))$

# $\beta$ -Reduction

Recall that  $\lambda x.M$  is intended to represent the function  $f$  such that  $f(x) = M$  for all  $x$ . We can regard  $\lambda x.M$  as a function on  $\lambda$ -terms via substitution: map each  $N$  to  $M[N/x]$ .

So the natural notion of computation for  $\lambda$ -terms is given by stepping from a

$\beta$ -redex  $(\lambda x.M)N$

to the corresponding

$\beta$ -reduct  $M[N/x]$

# Substitution $N[M/x]$

$$\begin{aligned}x[M/x] &= M \\y[M/x] &= y \quad \text{if } y \neq x \\(\lambda y.N)[M/x] &= \lambda y.N[M/x] \quad \text{if } y \# (M x) \\(N_1 N_2)[M/x] &= N_1[M/x] N_2[M/x]\end{aligned}$$

$N[M/x]$  = result of replacing all free occurrences of  $x$  in  $N$  with  $M$ , avoiding "capture" of free variables in  $M$  by  $\lambda$ -binders in  $N$

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Side-condition  $y \# (M x)$  ( $y$  does not occur in  $M$  and  $y \neq x$ ) makes substitution “capture-avoiding”.

E.g. if  $x \neq y$

$$(\lambda y.x)[y/x] \neq \lambda y.y$$

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Can always satisfy this  up to  $\alpha$ -equivalence

E.g. if  $x \neq y \neq z \neq x$

$$(\lambda y.x)[y/x] =_{\alpha} (\lambda z.x)[y/x] = \lambda z.y$$

In fact  $N \mapsto N[M/x]$  induces a totally defined function from the set of  $\alpha$ -equivalence classes of  $\lambda$ -terms to itself.

||

$$\lambda x. (\lambda z. z) y x \left[ \lambda z. y / y \right]$$

=

$$\lambda x. (\lambda z. z) y x \quad [ \lambda z. y / y ]$$

no possible  
capture

$$\lambda x. (\lambda z. z) y x \left[ \lambda z. y / y \right]$$
$$= \lambda x. (\lambda z. z) (\lambda z. y) x$$

---

$$\lambda x. (\lambda u. u) x y \left[ \lambda y. x / y \right]$$
$$=$$



$$\lambda x. (\lambda z. z) y x \quad [ \lambda x. y / y ]$$
$$= \lambda x. (\lambda z. z) (\lambda x. y) x$$

---

$$\lambda x. (\lambda u. u) x y \quad [ \lambda y. x / y ] \quad \text{possible capture}$$
$$=$$

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---

$$\lambda x. (\lambda u. u) x y \ [ \ \lambda y. x / y \ ] \quad \text{possible capture...}$$

$$=_{\alpha} \lambda z. (\lambda u. u) z y \ [ \ \lambda y. x / y \ ] \quad \text{...}\alpha\text{-convert to avoid}$$

$$\lambda x. (\lambda z. z) y x \left[ \lambda x. y / y \right]$$

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$$\lambda x. (\lambda u. u) x y \left[ \lambda y. x / y \right]$$

possible capture...

$$\stackrel{\alpha}{=} \lambda z. (\lambda u. u) z y \left[ \lambda y. x / y \right]$$

... $\alpha$ -convert to avoid

$$= \lambda z. (\lambda u. u) z (\lambda y. x)$$

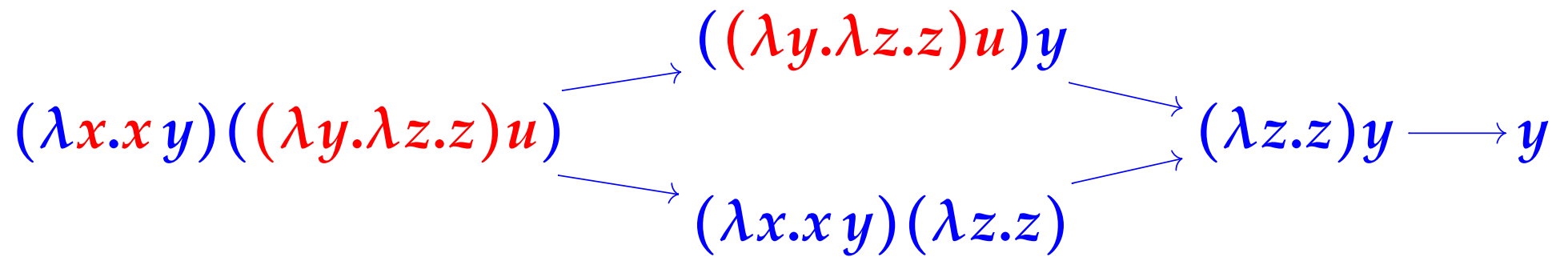
# $\beta$ -Reduction

One-step  $\beta$ -reduction,  $M \rightarrow M'$ :

$$\frac{}{(\lambda x.M)N \rightarrow M[N/x]} \qquad \frac{M \rightarrow M'}{\lambda x.M \rightarrow \lambda x.M'}$$
$$\frac{M \rightarrow M'}{MN \rightarrow M'N} \qquad \frac{M \rightarrow M'}{NM \rightarrow NM'}$$
$$\frac{N =_{\alpha} M \quad M \rightarrow M' \quad M' =_{\alpha} N'}{N \rightarrow N'}$$

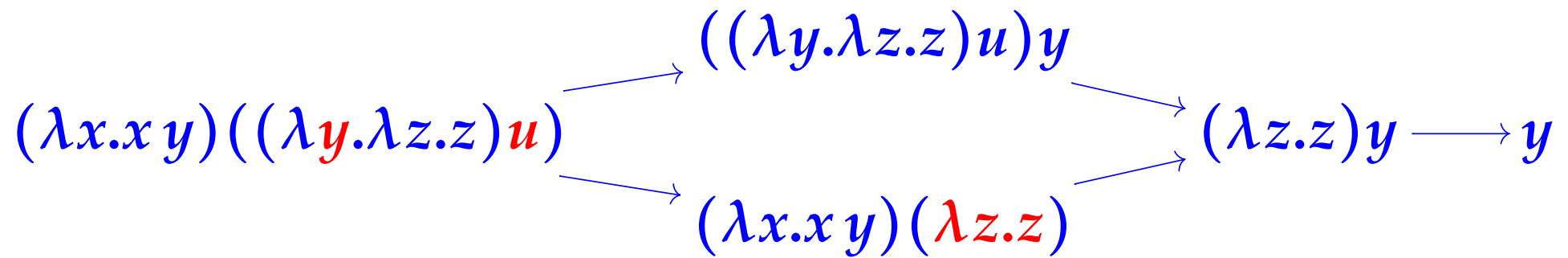
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E.g.



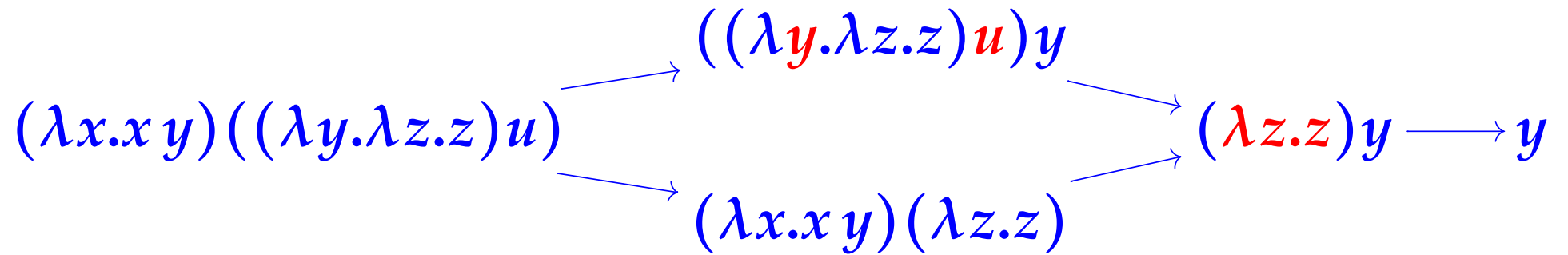
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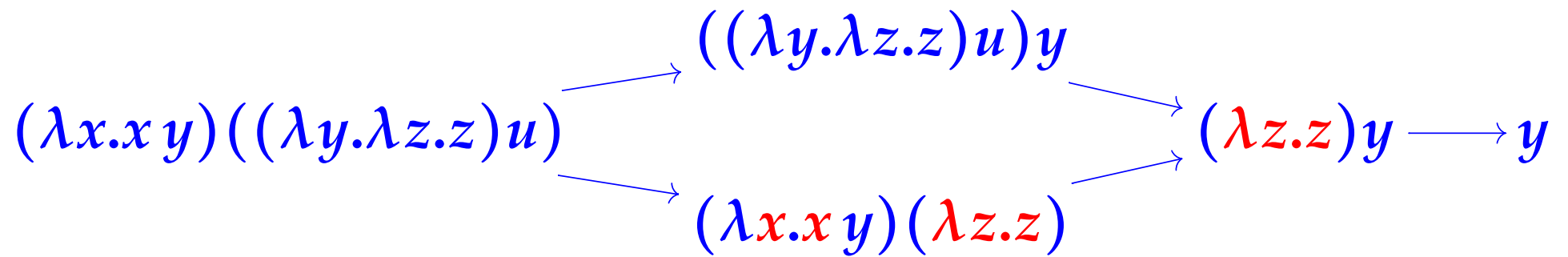
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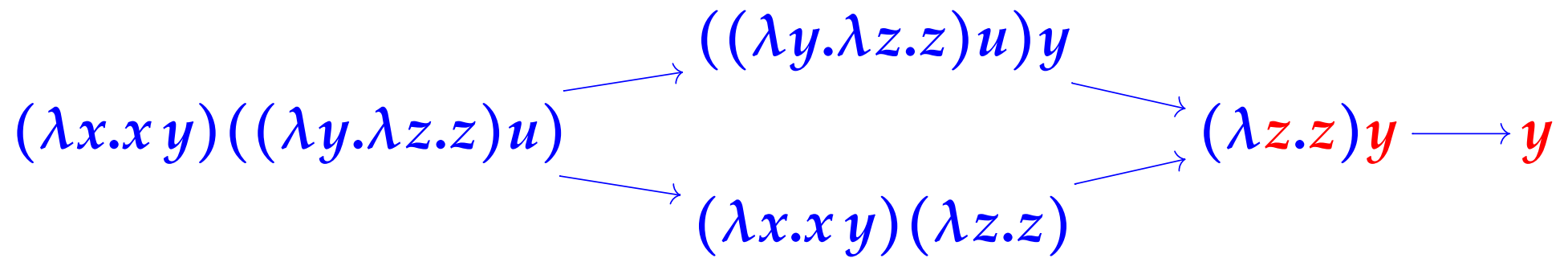
E.g.





# $\beta$ -Reduction

E.g.



Many-step  $\beta$ -reduction,  $M \rightarrow\!\!\rightarrow M'$ :

$$\frac{M =_{\alpha} M'}{M \rightarrow\!\!\rightarrow M'} \quad \left[ \frac{M \rightarrow M'}{M \rightarrow\!\!\rightarrow M'} \right] \quad \frac{M \rightarrow\!\!\rightarrow M' \quad M' \rightarrow M''}{M \rightarrow\!\!\rightarrow M''}$$

(no steps)                      (1 step)                      (1 more step)

E.g.

$$(\lambda x. x y) ((\lambda y z. z) u) \rightarrow\!\!\rightarrow y$$

# $\beta$ -Conversion $M =_{\beta} N$

Informally:  $M =_{\beta} N$  holds if  $N$  can be obtained from  $M$  by performing zero or more steps of  $\alpha$ -equivalence,  $\beta$ -reduction, or  $\beta$ -expansion (= inverse of a reduction).

E.g.  $u((\lambda x y. v x)y) =_{\beta} (\lambda x. u x)(\lambda x. v y)$

because  $(\lambda x. u x)(\lambda x. v y) \rightarrow u(\lambda x. v y)$

and so we have

$u((\lambda x y. v x)y) =_{\alpha} u((\lambda x y'. v x)y)$

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# $\beta$ -Conversion $M =_{\beta} N$

is the binary relation inductively generated by the rules:

$$\frac{M =_{\alpha} M'}{M =_{\beta} M'}$$

$$\frac{M \rightarrow M'}{M =_{\beta} M'}$$

$$\frac{M =_{\beta} M'}{M' =_{\beta} M}$$

$$\frac{M =_{\beta} M' \quad M' =_{\beta} M''}{M =_{\beta} M''}$$

$$\frac{M =_{\beta} M'}{\lambda x.M =_{\beta} \lambda x.M'}$$

$$\frac{M =_{\beta} M' \quad N =_{\beta} N'}{MN =_{\beta} M'N'}$$

# Church-Rosser Theorem

**Theorem.**  $\rightarrow$  is **confluent**, that is, if  $M_1 \leftarrow M \rightarrow M_2$ , then there exists  $M'$  such that  $M_1 \rightarrow M' \leftarrow M_2$ .

[Proof omitted.]



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**Corollary.** To show that two terms are  $\beta$ -convertible, it suffices to show that they both reduce to the same term. More precisely:  $M_1 =_{\beta} M_2$  iff  $\exists M (M_1 \rightarrow M \leftarrow M_2)$ .

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**Corollary.**  $M_1 =_{\beta} M_2$  iff  $\exists M (M_1 \rightarrow M \leftarrow M_2)$ .

**Proof.**  $=_{\beta}$  satisfies the rules generating  $\rightarrow$ ; so  $M \rightarrow M'$  implies  $M =_{\beta} M'$ . Thus if  $M_1 \rightarrow M \leftarrow M_2$ , then  $M_1 =_{\beta} M =_{\beta} M_2$  and so  $M_1 =_{\beta} M_2$ .

Conversely,

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Conversely, the relation  $\{(M_1, M_2) \mid \exists M (M_1 \rightarrow M \leftarrow M_2)\}$  satisfies the rules generating  $=_{\beta}$ : the only difficult case is closure of the relation under transitivity and for this we use the Church-Rosser theorem:  $M_1 \rightarrow M \leftarrow M_2 \rightarrow M' \leftarrow M_3$

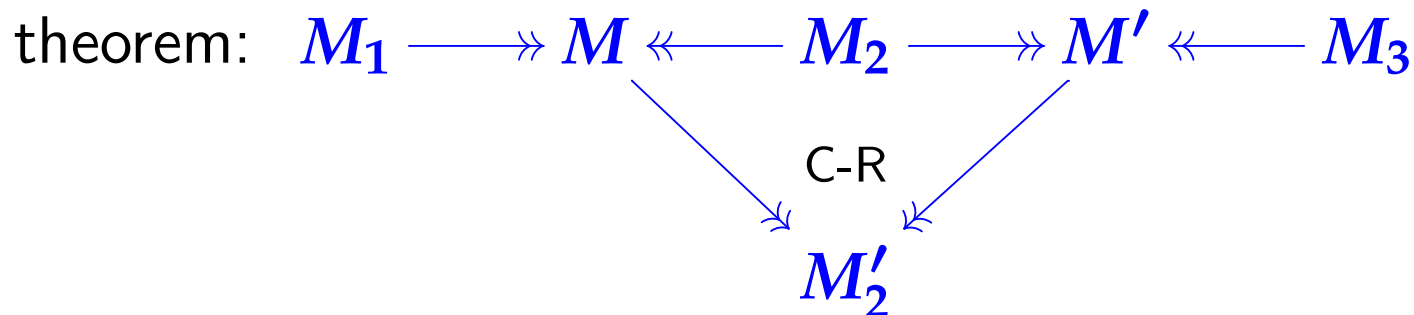
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# $\beta$ -Normal Forms

**Definition.** A  $\lambda$ -term  $N$  is in  $\beta$ -normal form (nf) if it contains no  $\beta$ -redexes (no sub-terms of the form  $(\lambda x.M)M'$ ).  $M$  has  $\beta$ -nf  $N$  if  $M =_{\beta} N$  with  $N$  a  $\beta$ -nf.

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Note that if  $N$  is a  $\beta$ -nf and  $N \rightarrow N'$ , then it must be that  $N =_{\alpha} N'$  (why?).

Hence if  $N_1 =_{\beta} N_2$  with  $N_1$  and  $N_2$  both  $\beta$ -nfs, then  $N_1 =_{\alpha} N_2$ . (For if  $N_1 =_{\beta} N_2$ , then by Church-Rosser  $N_1 \rightarrow M' \leftarrow N_2$  for some  $M'$ , so  $N_1 =_{\alpha} M' =_{\alpha} N_2$ .)

**So the  $\beta$ -nf of  $M$  is unique up to  $\alpha$ -equivalence if it exists.**

# Non-termination

Some  $\lambda$  terms have no  $\beta$ -nf.

E.g.  $\Omega \triangleq (\lambda x.x x)(\lambda x.x x)$  satisfies

- ▶  $\Omega \rightarrow (x x)[(\lambda x.x x)/x] = \Omega$ ,
- ▶  $\Omega \rightarrow M$  implies  $\Omega =_{\alpha} M$ .

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So there is no  $\beta$ -nf  $N$  such that  $\Omega =_{\beta} N$ .

**A term can possess both a  $\beta$ -nf and infinite chains of reduction from it.**

E.g.  $(\lambda x.y)\Omega \rightarrow y$ , but also  $(\lambda x.y)\Omega \rightarrow (\lambda x.y)\Omega \rightarrow \dots$ .

# Non-termination

**Normal-order reduction** is a deterministic strategy for reducing  $\lambda$ -terms: reduce the “left-most, outer-most” redex first.

- ▶ left-most: reduce  $M$  before  $N$  in  $M N$ , and then
- ▶ outer-most: reduce  $(\lambda x.M)N$  rather than either of  $M$  or  $N$ .

(cf. call-by-name evaluation).

**Fact:** normal-order reduction of  $M$  always reaches the  $\beta$ -nf of  $M$  if it possesses one.

$$\frac{M_1 =_{\alpha} M'_1 \quad M'_1 \rightarrow_{\text{NOR}} M'_2 \quad M'_2 =_{\alpha} M_2}{M_1 \rightarrow_{\text{NOR}} M_2}$$

$$\frac{M \rightarrow_{\text{NOR}} M'}{\lambda x. M \rightarrow_{\text{NOR}} \lambda x. M'}$$

$$\frac{M_1 \rightarrow_{\text{NOR}} M'_1}{M_1 M_2 \rightarrow_{\text{NOR}} M'_1 M_2}$$

$$\frac{}{(\lambda x. M) M' \rightarrow_{\text{NOR}} M[M'/x]}$$

$$\frac{M \rightarrow_{\text{NOR}} M'}{UM \rightarrow_{\text{NOR}} UM'}$$

where

$$\begin{cases} U ::= x \mid UN \\ N ::= \lambda x. N \mid U \end{cases}$$

$\beta$ -normal forms

"neutral" forms