# VI. Approximation Algorithms: Travelling Salesman Problem

Thomas Sauerwald

Easter 2017

UNIVERSITY OF CAMBRIDGE

# Outline

Introduction

General TSP

Metric TSP

# The Traveling Salesman Problem (TSP)

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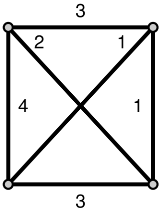


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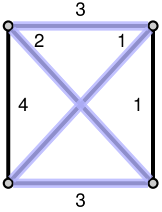


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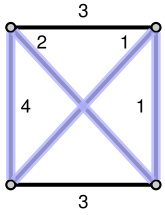


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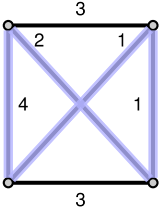


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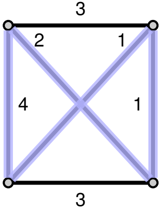


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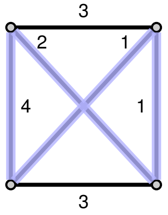


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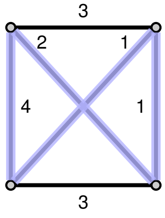


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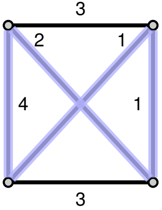


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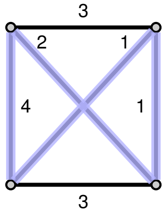


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Even this version is NP hard (Ex. 35.2-2)

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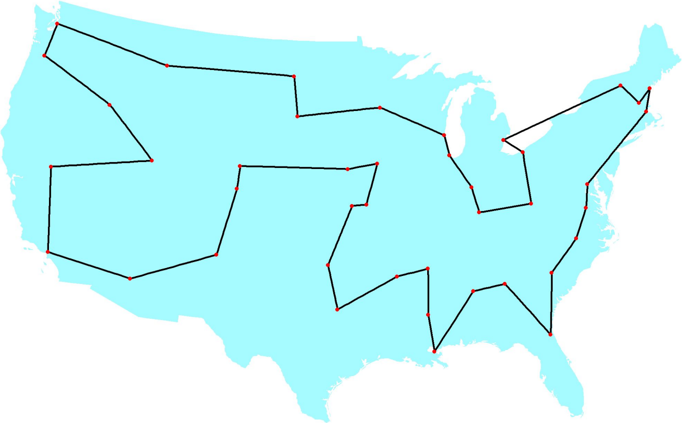
# History of the TSP problem (1954)

Dantzig, Fulkerson and Johnson found an optimal tour through 42 cities.



**http://www.math.uwaterloo.ca/tsp/history/img/dantzig_big.html**

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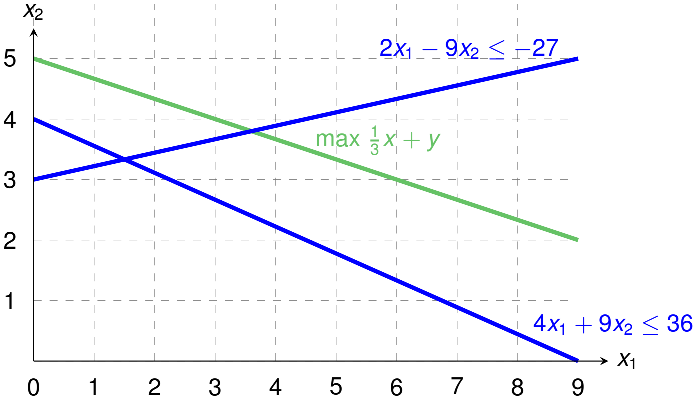

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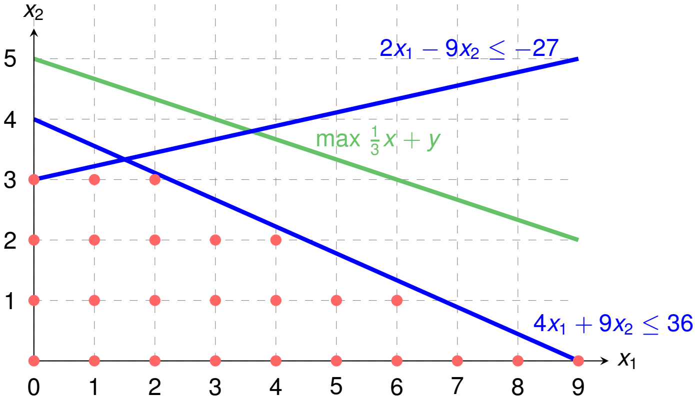

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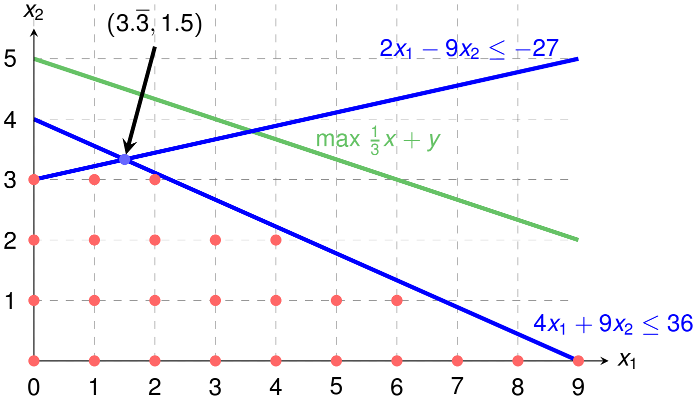

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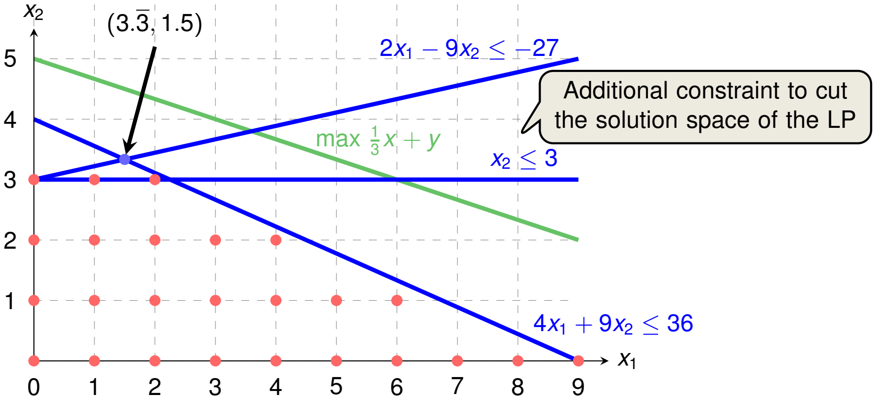

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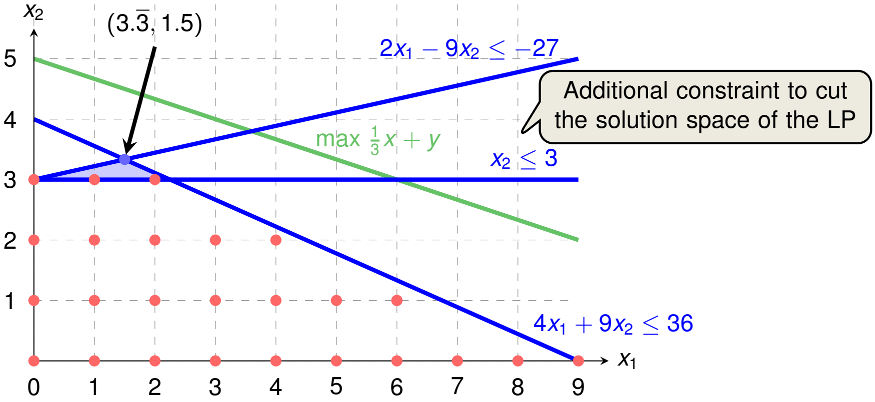

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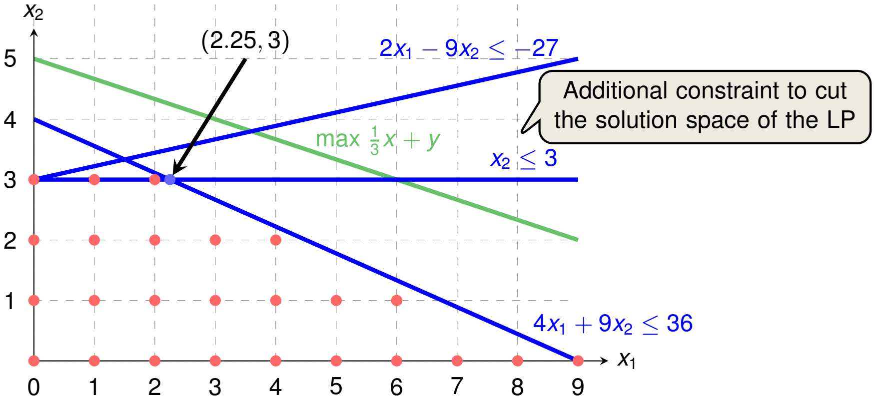

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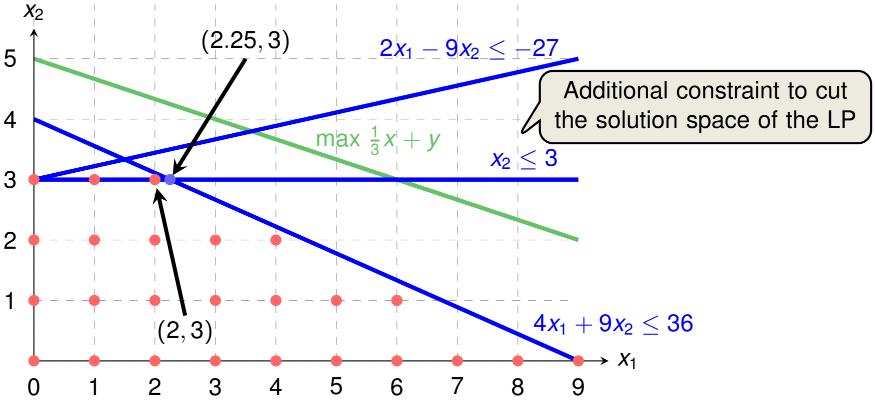

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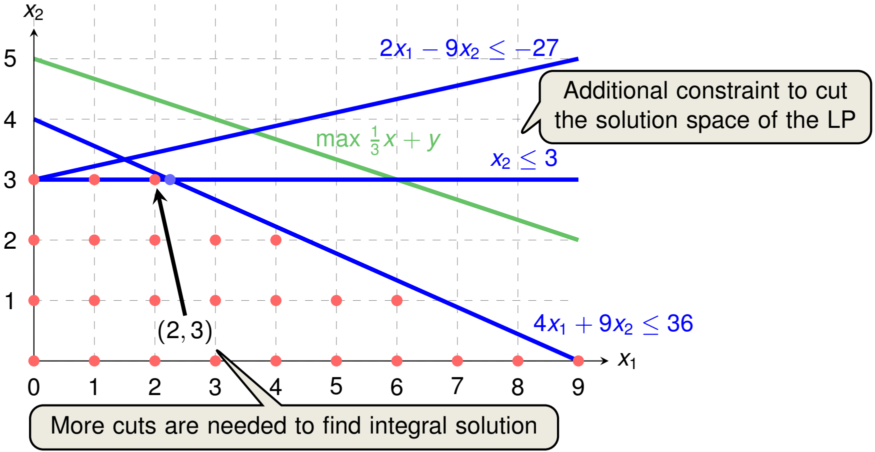

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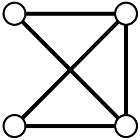

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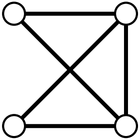

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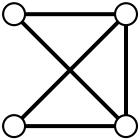

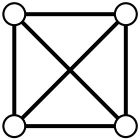

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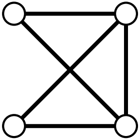
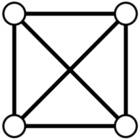
$G = (V, E)$ 

 $G' = (V, E')$

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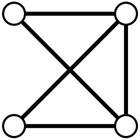

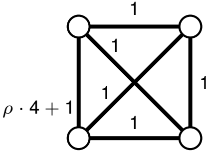


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Large weight will render this edge useless!

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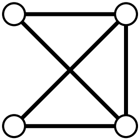

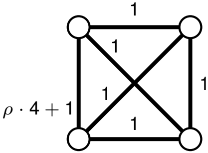


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Can create representations of $G'$ and $c$ in time polynomial in $|V|$ and $|E|$!

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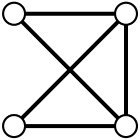

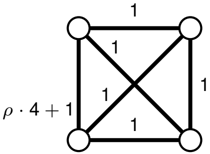


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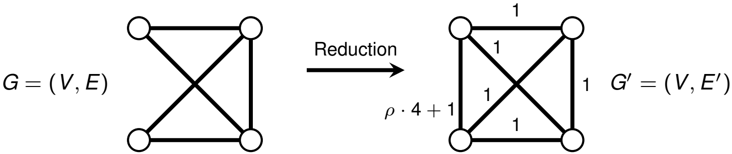

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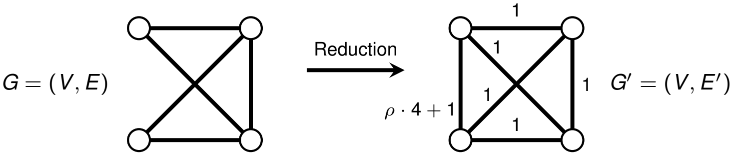

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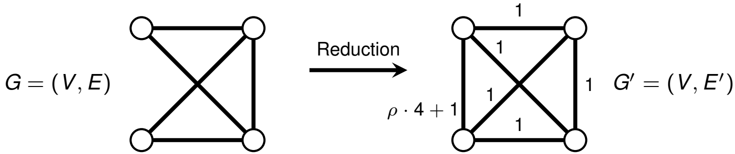

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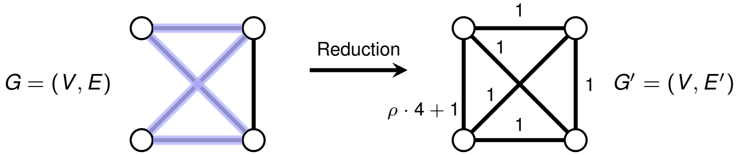

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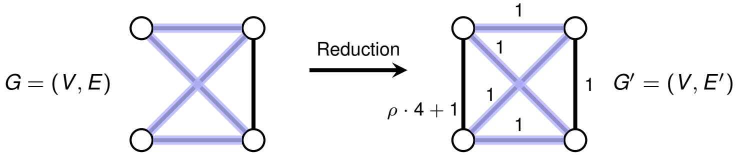

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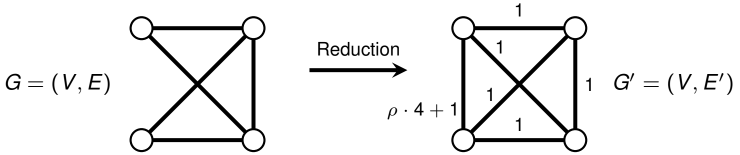

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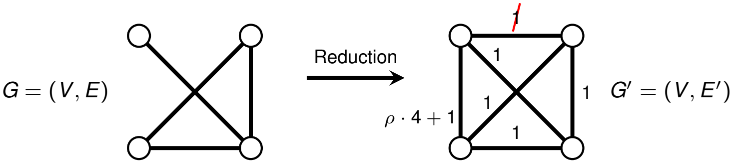

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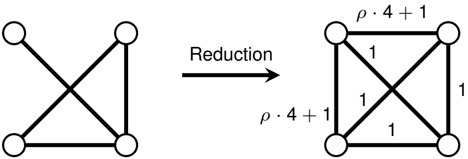


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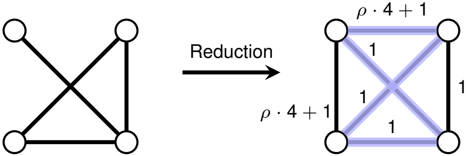


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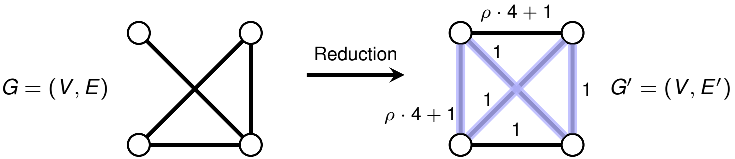

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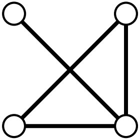

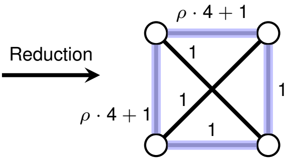


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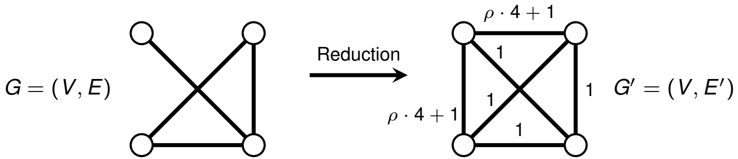

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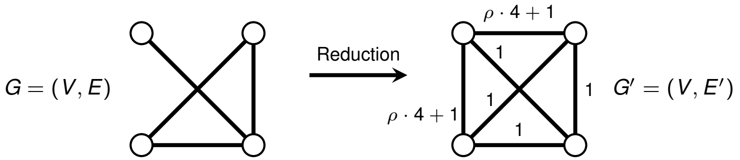

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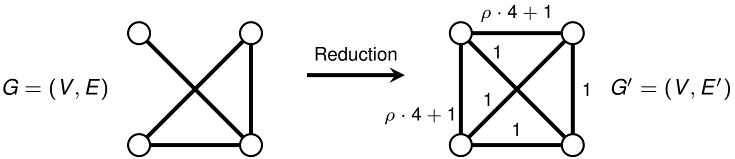

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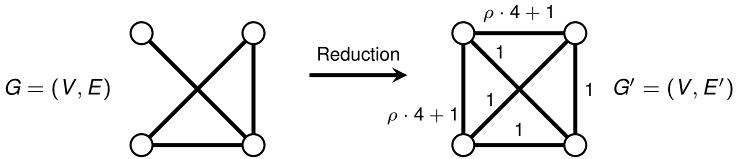

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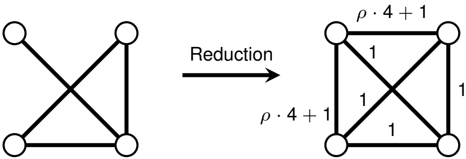


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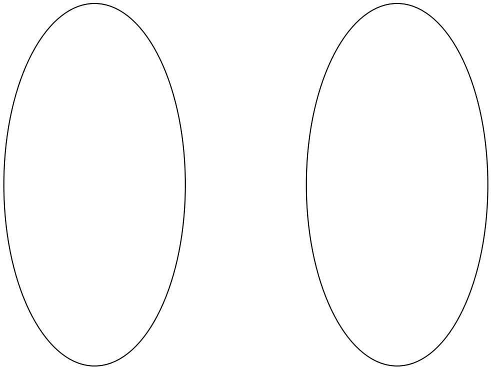

instances of Hamilton          instances of TSP

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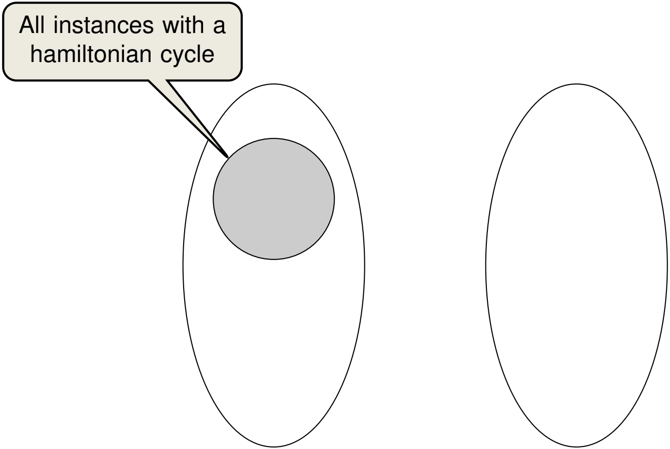


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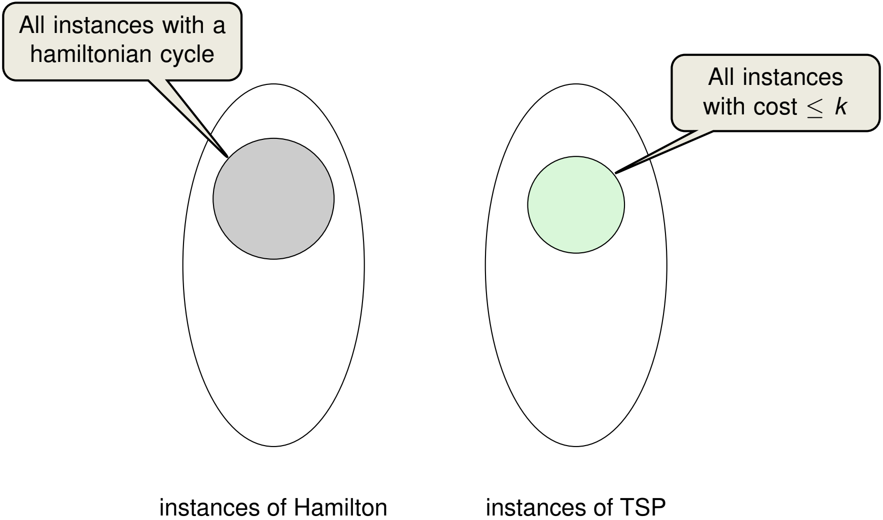

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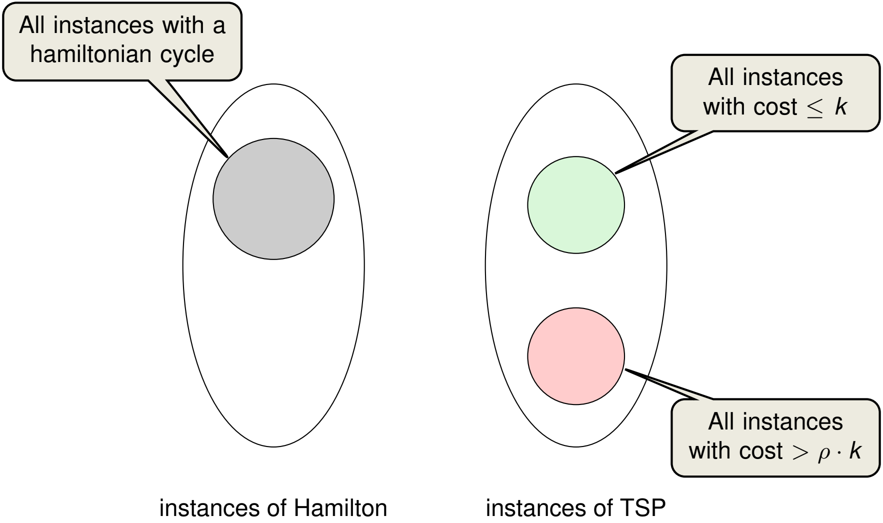

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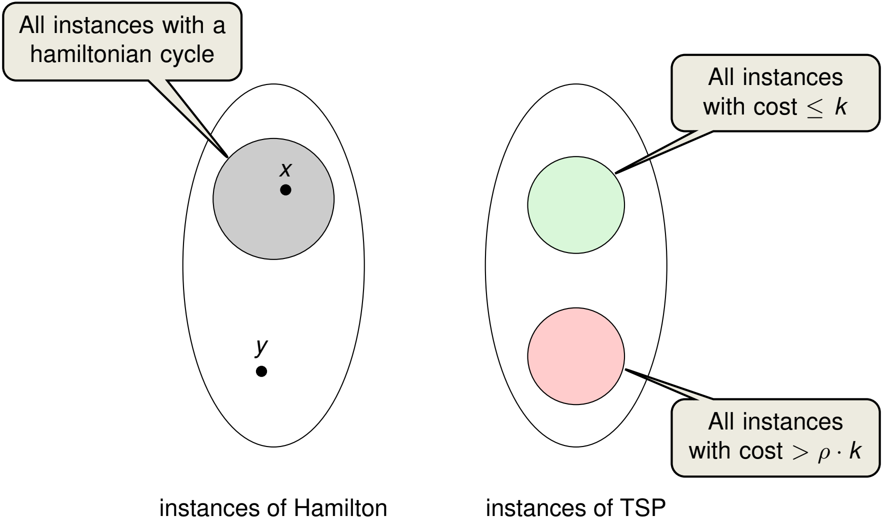

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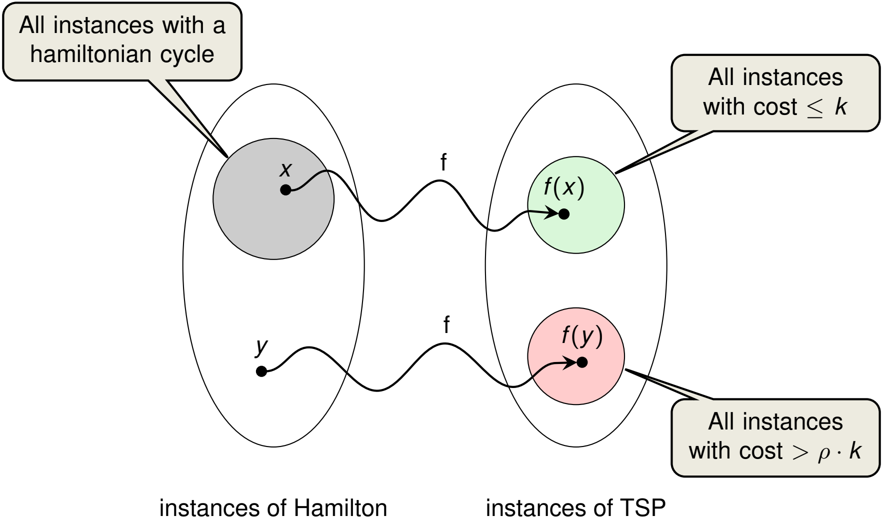

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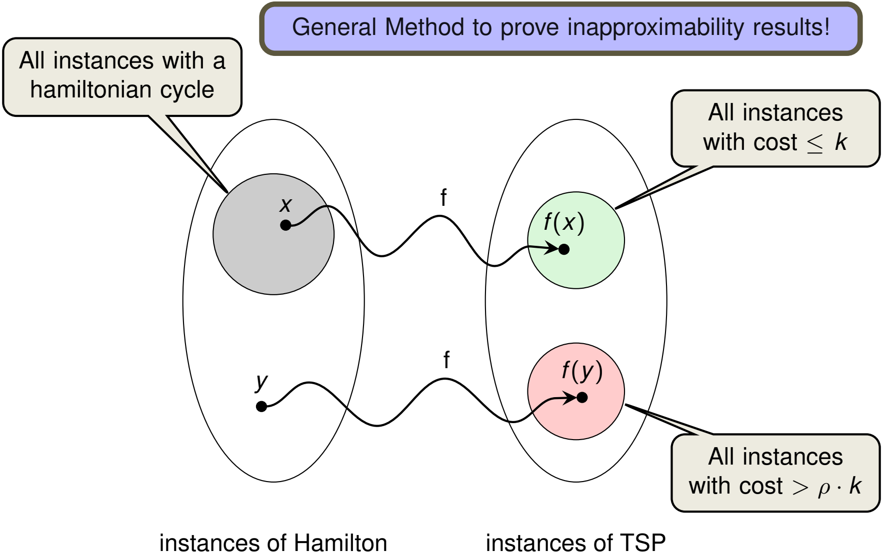

# Outline

Introduction

General TSP

Metric TSP

# Metric TSP (TSP Problem with the Triangle Inequality)

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Runtime is dominated by MST-PRIM, which is $\Theta(V^2)$.

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Remember: In the Metric-TSP problem, $G$ is a complete graph.

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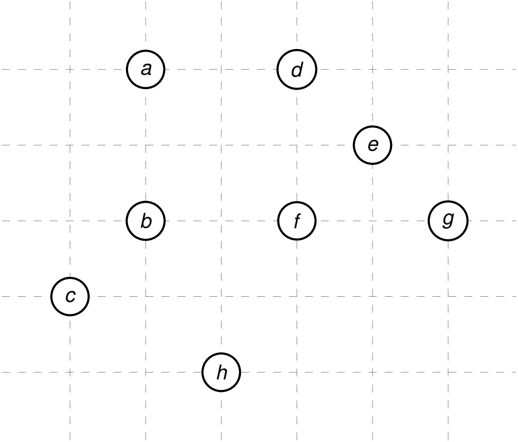

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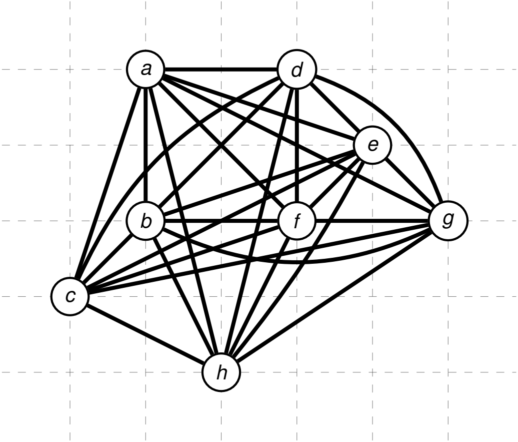


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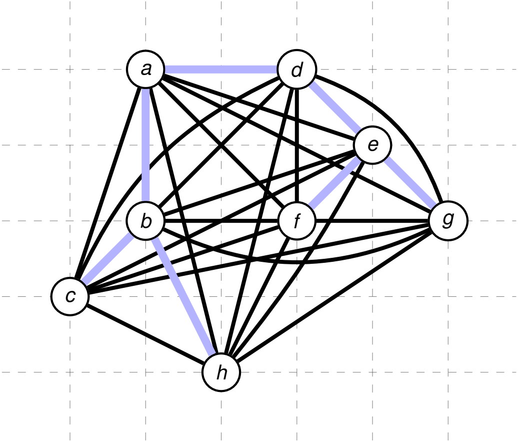


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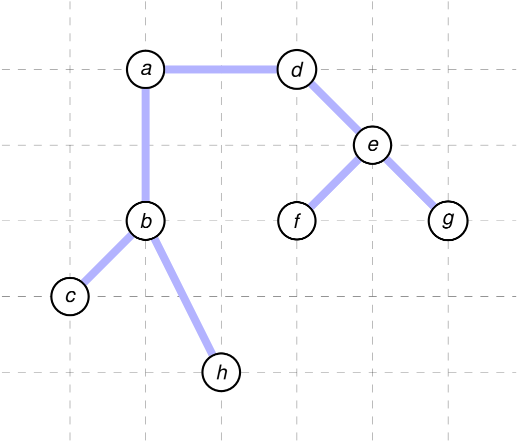


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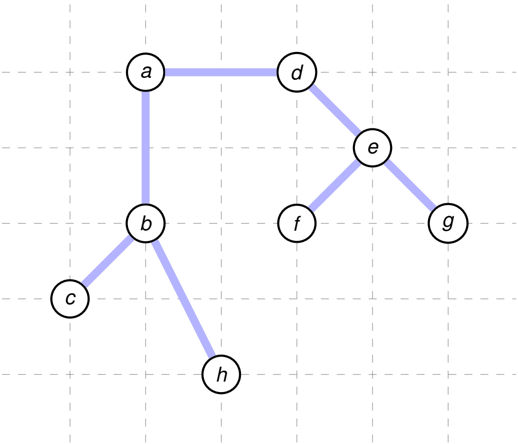


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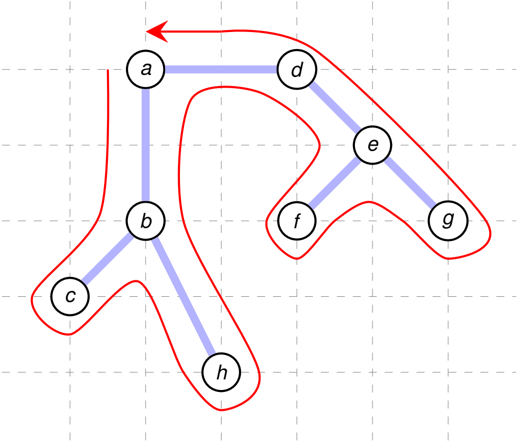


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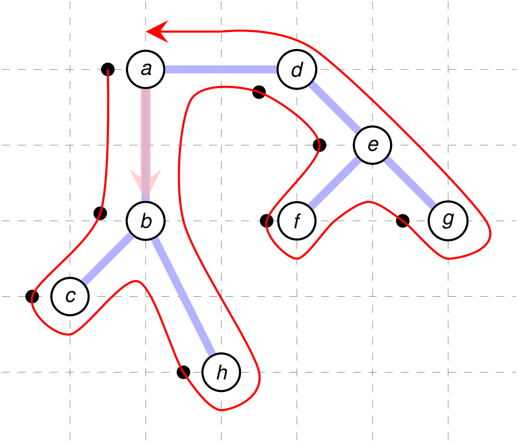


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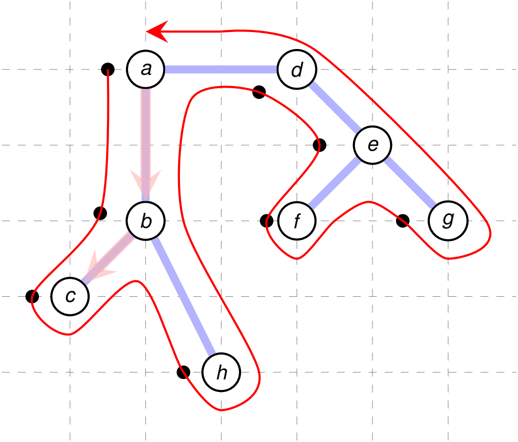


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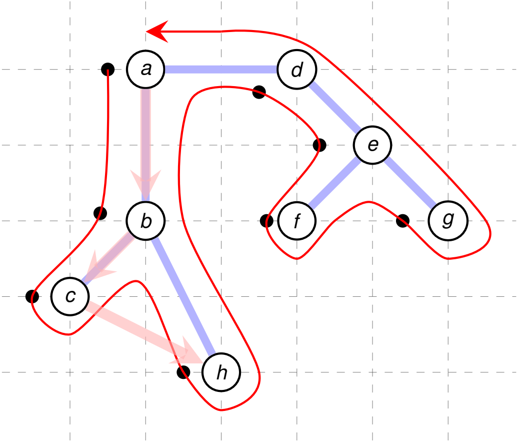


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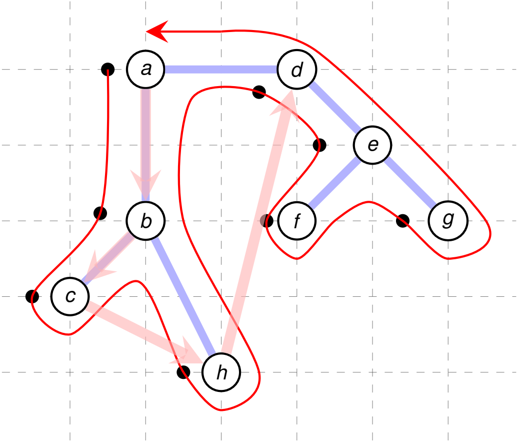


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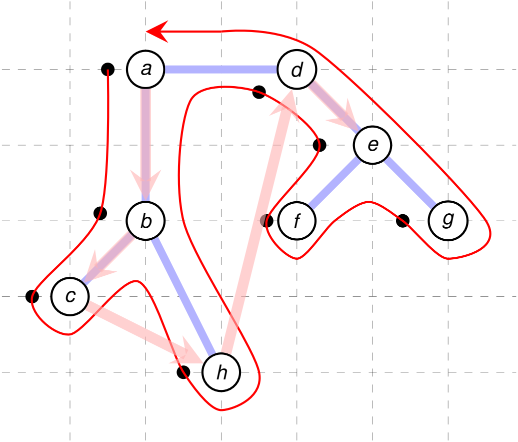


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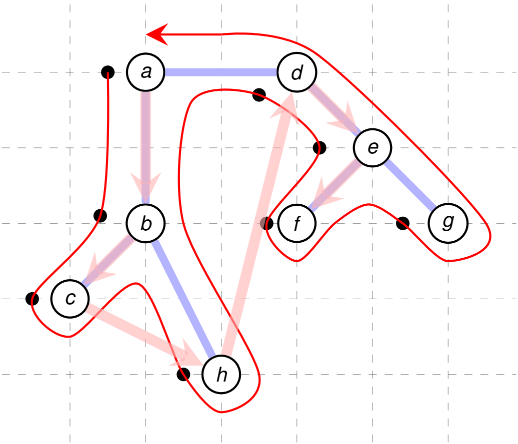


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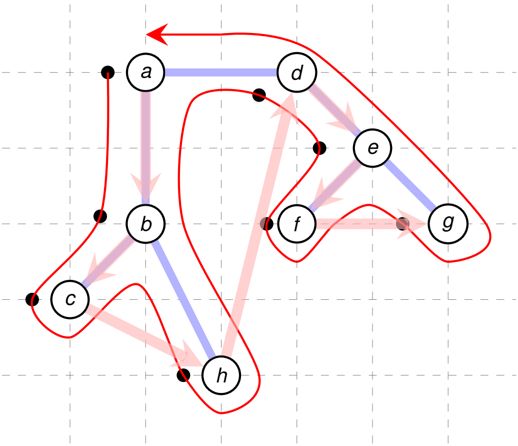
2. Perform preorder walk on MST ✓
3. Return list of vertices according to the preorder tree walk

## Run of APPROX-TSP-TOUR

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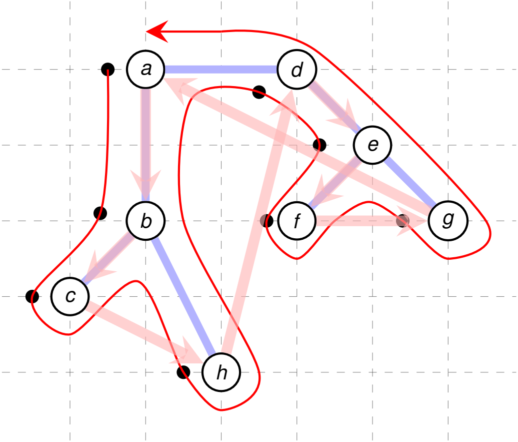


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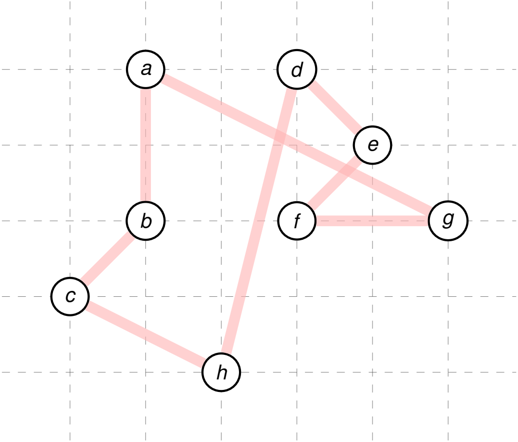


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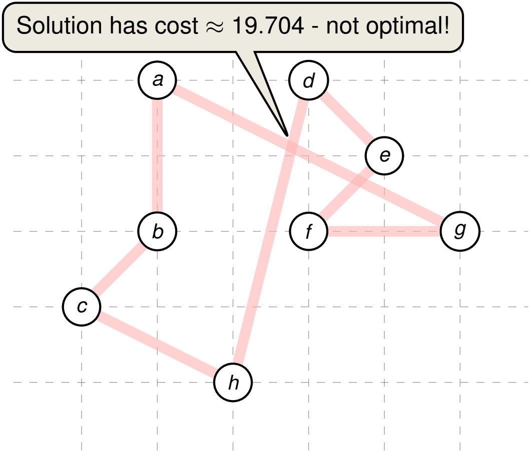


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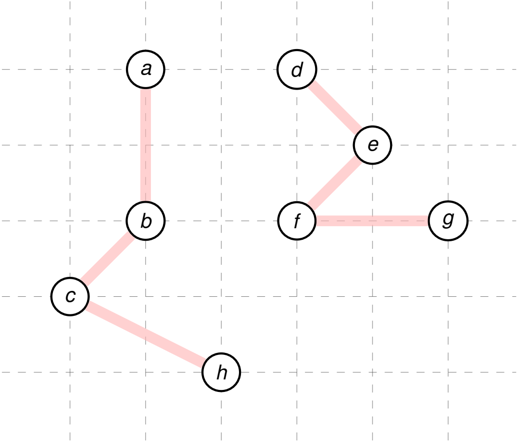


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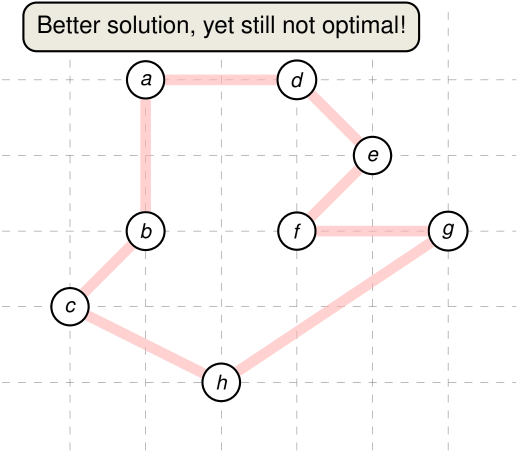


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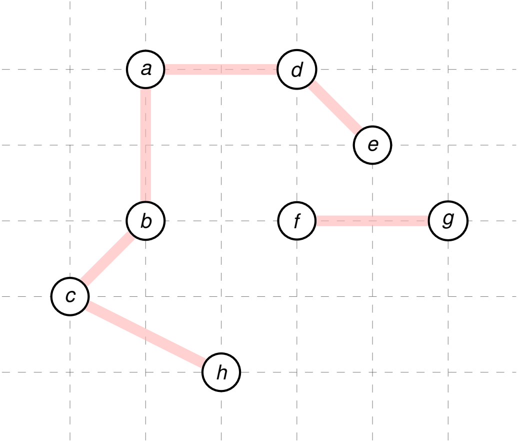


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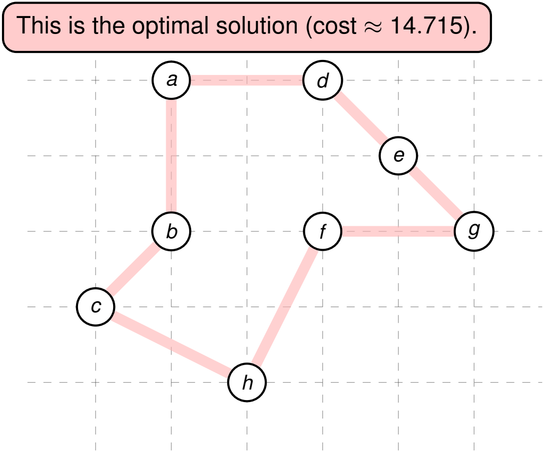


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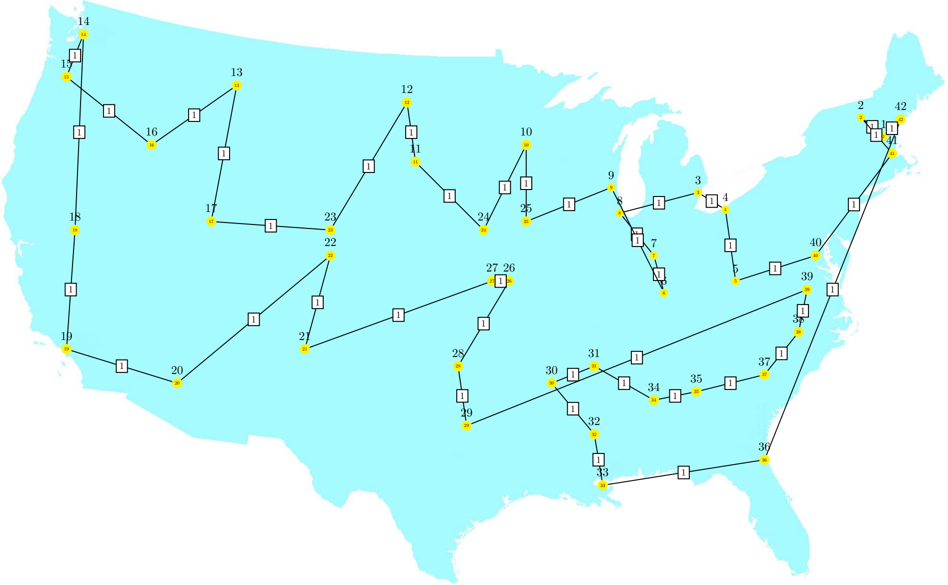
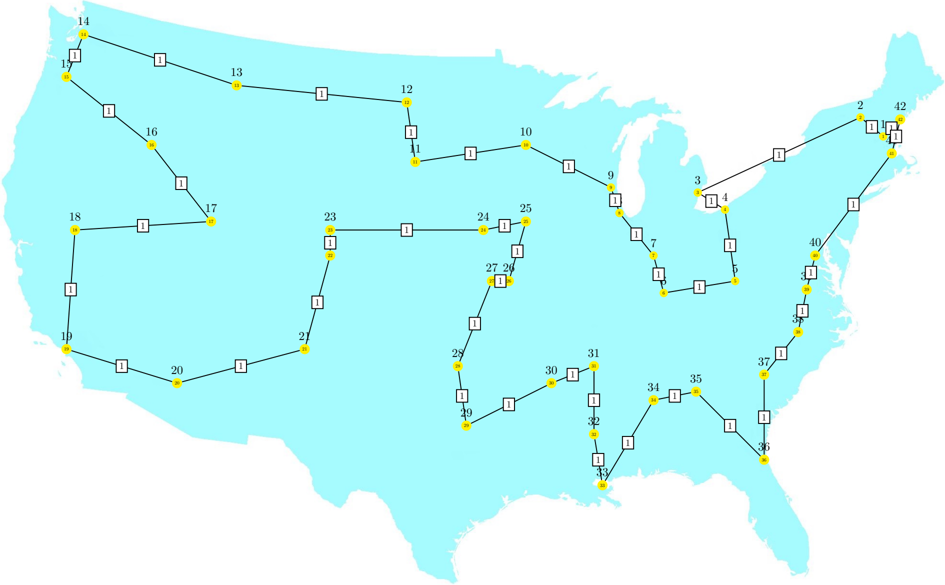
# Approximate Solution: Objective 921



# Optimal Solution: Objective 699



## Proof of the Approximation Ratio

**Theorem 35.2**

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

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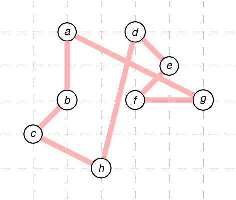


solution $H$ of APPROX-TSP

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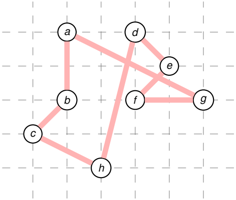


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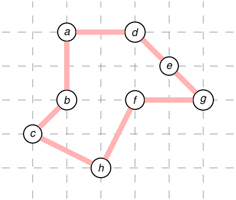


optimal solution $H^*$

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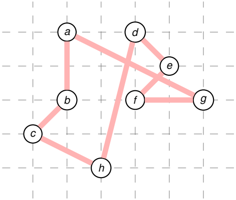


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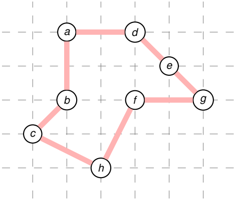


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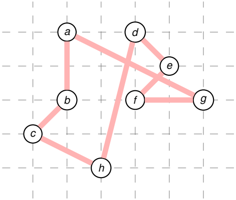

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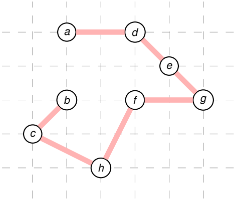

spanning tree $T$ as a subset of $H^*$

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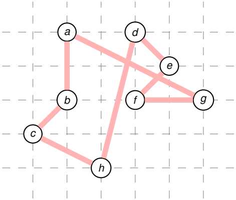


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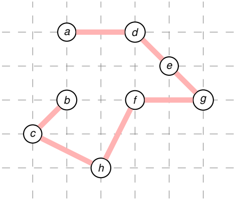


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Proof:

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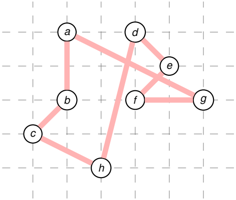
$\Rightarrow$ yields a spanning tree $T$ and $c(T) \leq c(H^*)$

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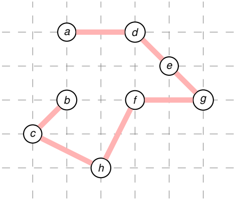


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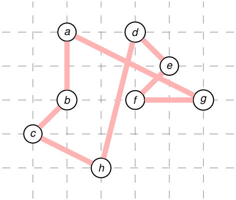
exploiting that all edge costs are non-negative!

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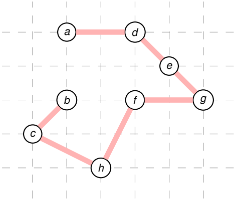


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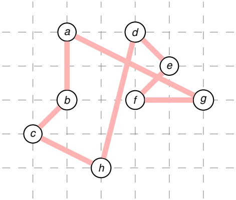


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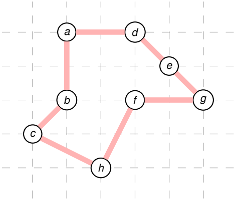


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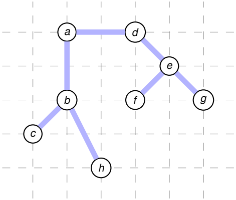
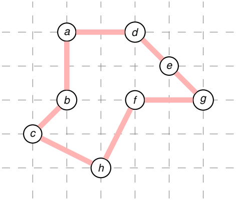



minimum spanning tree $T_{\min}$

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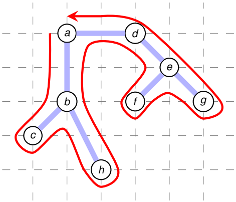
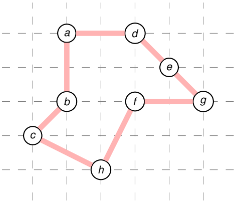



Walk $W = (a, b, c, b, h, b, a, d, e, f, e, g, e, d, a)$

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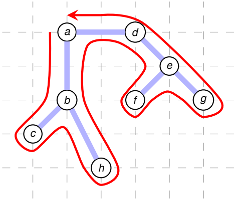


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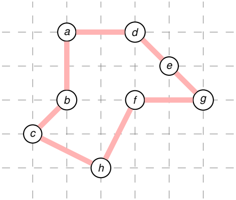


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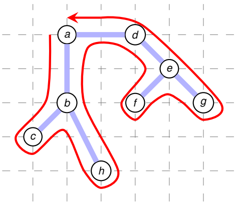


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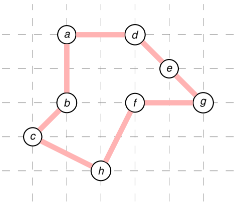


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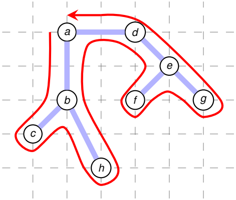


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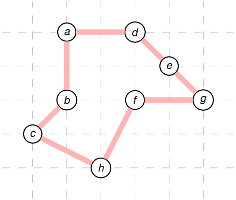


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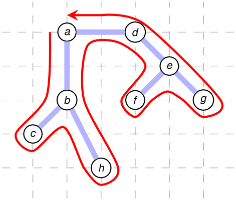


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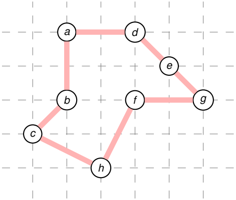


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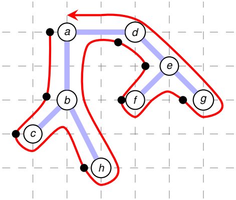


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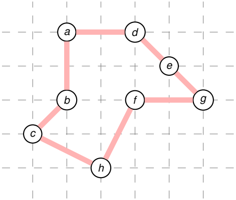


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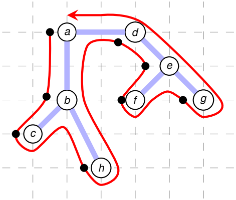
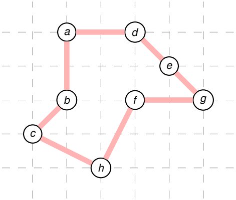



Walk $W = (a, b, c, \cancel{b}, h, \cancel{b}, \cancel{a}, d, e, f, \cancel{e}, g, \cancel{e}, \cancel{d}, a)$

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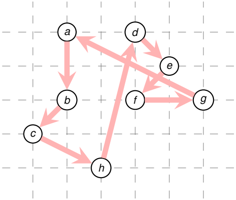
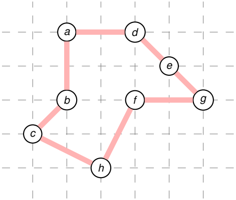



Tour $H = (a, b, c, h, d, e, f, g, a)$

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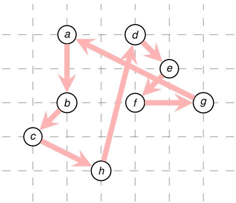
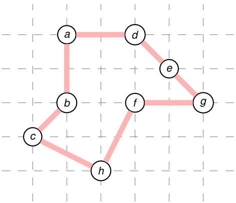
exploiting triangle inequality!

- Deleting duplicate vertices from $W$ yields a tour $H$ with smaller cost:




Tour $H = (a, b, c, h, d, e, f, g, a)$




optimal solution $H^*$

# Proof of the Approximation Ratio

**Theorem 35.2**

APPROX-TSP-TOUR is a polynomial-time 2-approximation for the traveling-salesman problem with the triangle inequality.

Proof:

- Consider the optimal tour $H^*$ and remove an arbitrary edge

$\Rightarrow$ yields a spanning tree $T$ and $c(T) \leq c(H^*)$

- Let $W$ be the full walk of the minimum spanning tree $T_{\min}$ (including repeated visits)

$\Rightarrow$ Full walk traverses every edge exactly twice, so

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exploiting triangle inequality!

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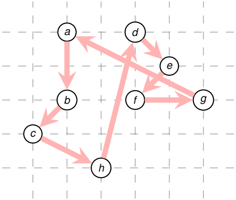


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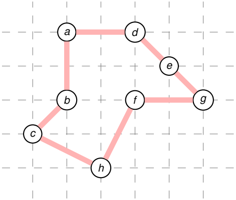


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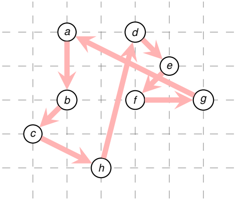


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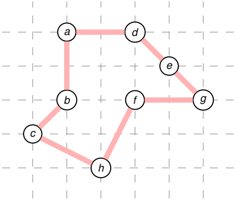


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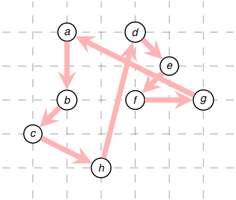


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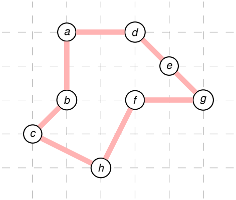


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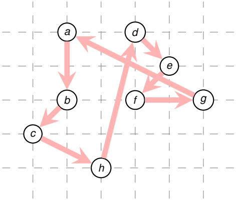


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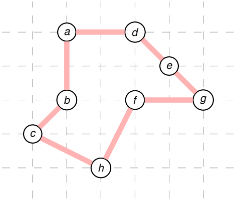


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# Christofides Algorithm

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Can we get a better approximation ratio?

CHRISTOFIDES$(G, c)$
select a vertex $r \in G.V$ to be a “root” vertex
compute a minimum spanning tree $T$ for $G$ from root $r$
    using MST-PRIM$(G, c, r)$
compute a perfect matching $M$ with minimum weight in the complete graph
    over the odd-degree vertices in $T$
let $H$ be a list of vertices, ordered according to when they are first visited
    in a Eulearian circuit of $T \cup M$
**return** $H$

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return H
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**Theorem (Christofides'76)**

There is a polynomial-time $\frac{3}{2}$-approximation algorithm for the travelling salesman problem with the triangle inequality.

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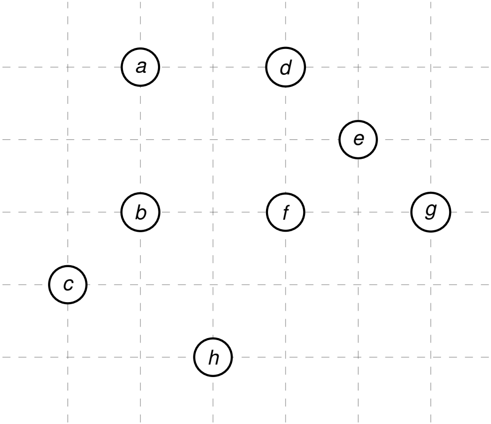

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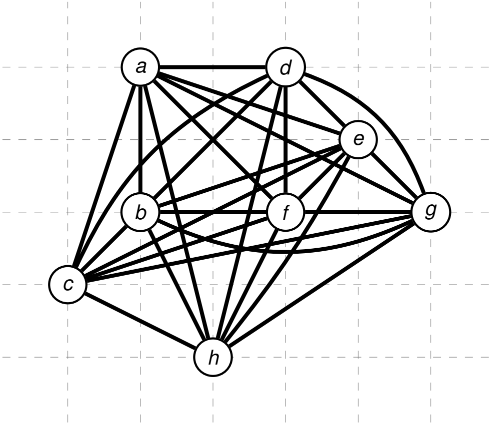
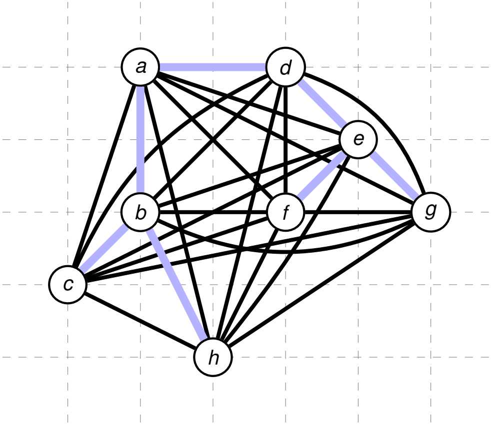



1. Compute MST

## Run of CHRISTOFIDES

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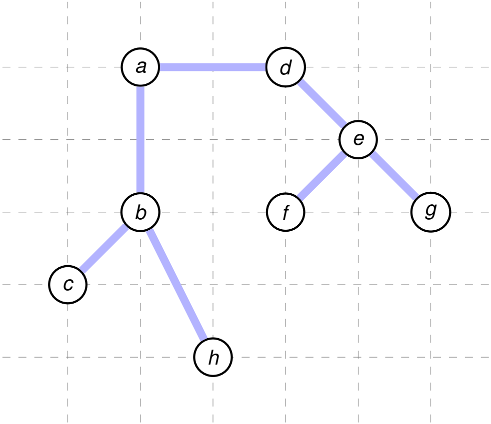


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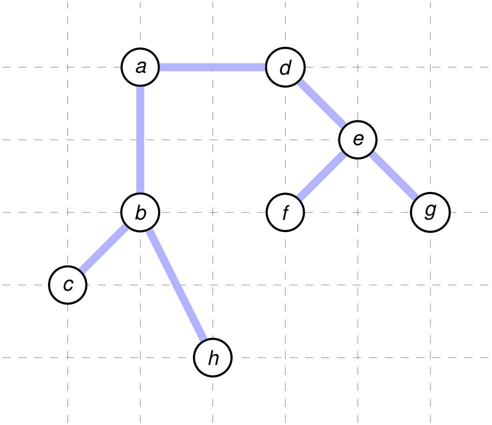


1. Compute MST ✓
2. Add a minimum-weight perfect matching $M$ of the odd vertices in $T$

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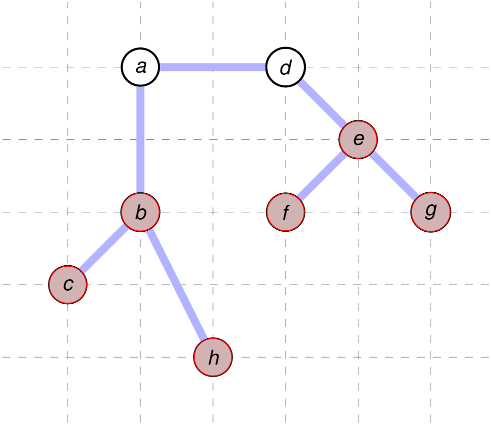


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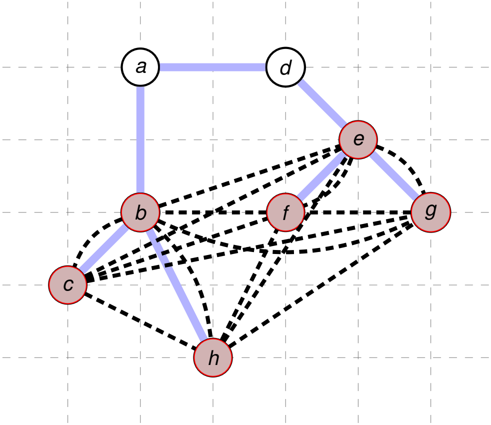


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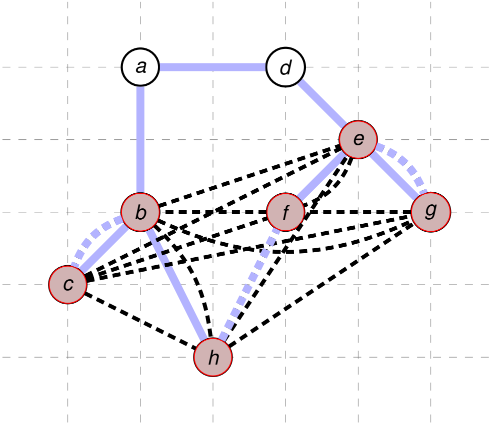


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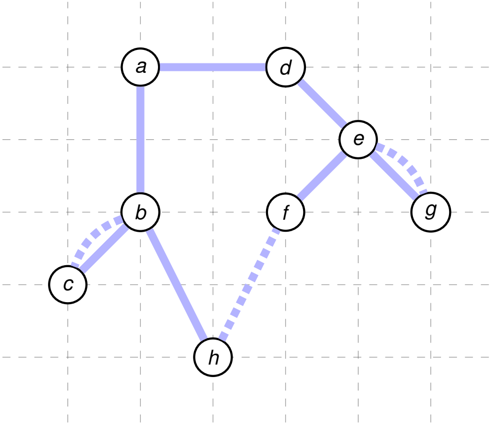


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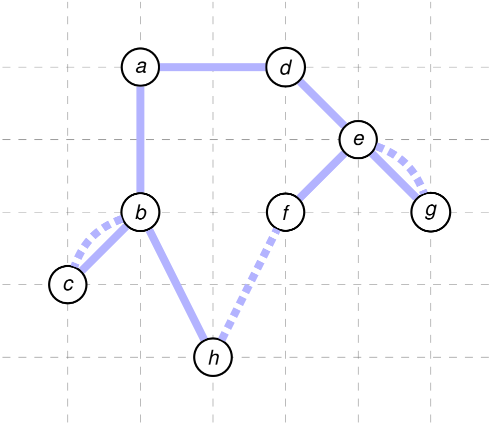


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3. Find an Eulerian Circuit

All vertices in $T \cup M$ have even degree!

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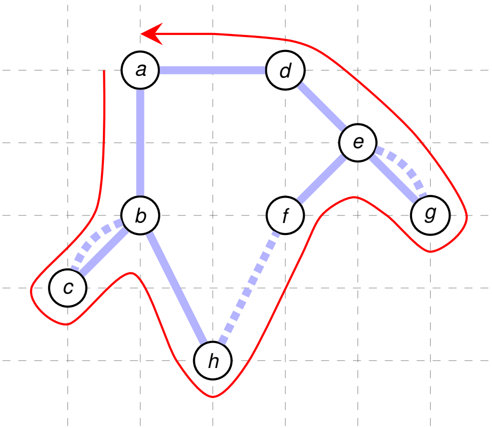


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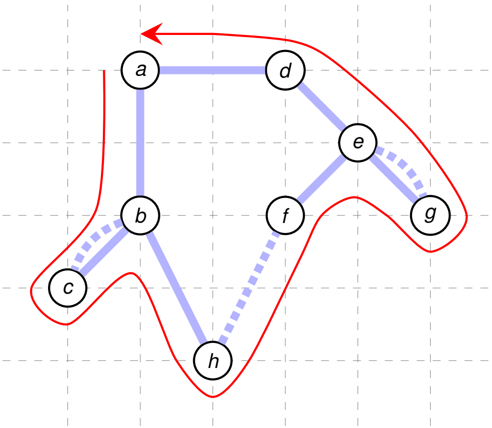


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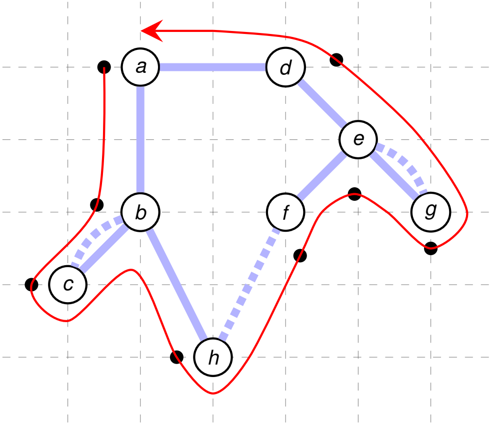


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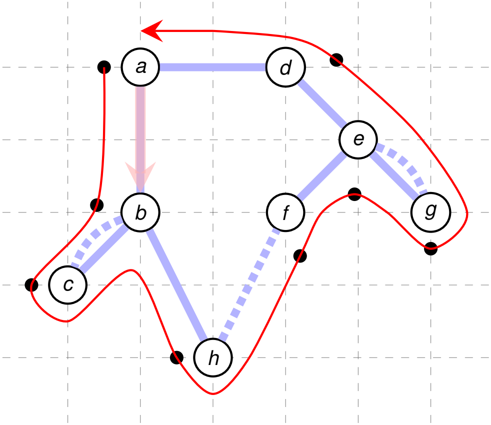


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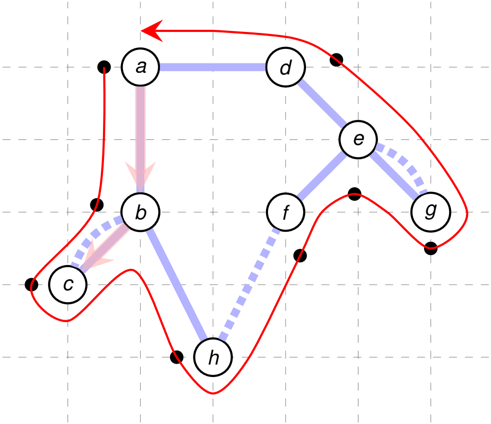


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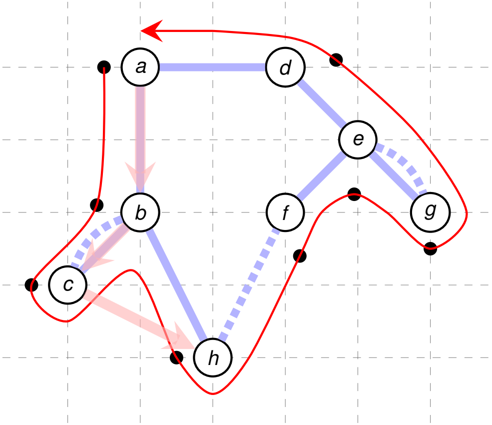


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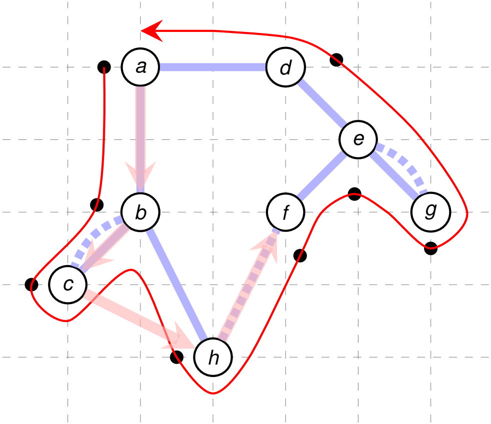


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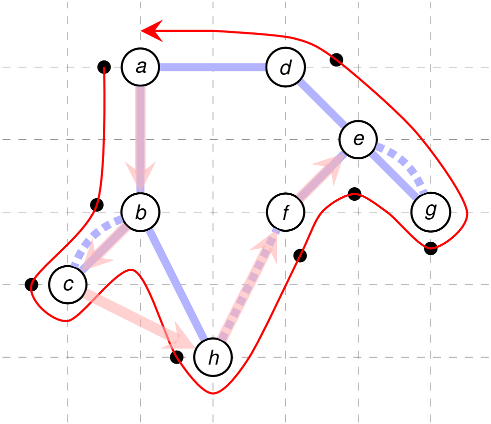


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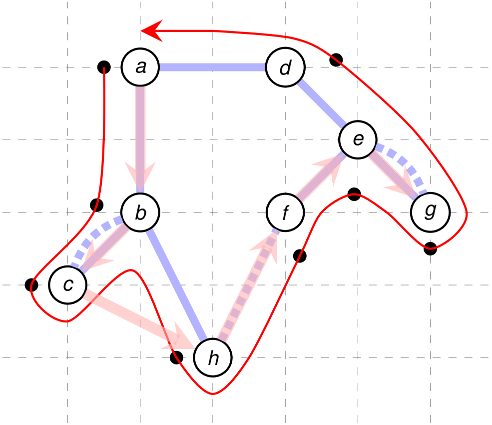


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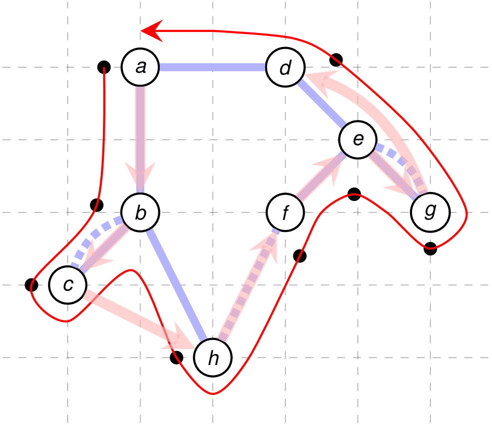


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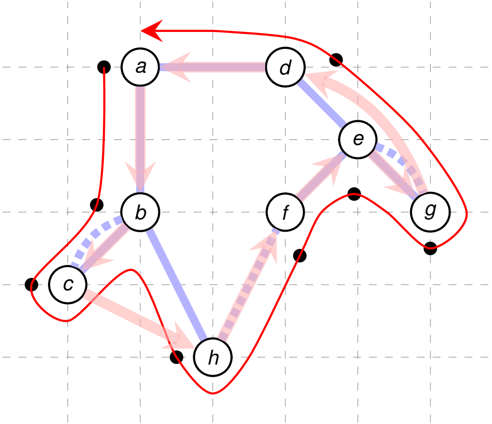


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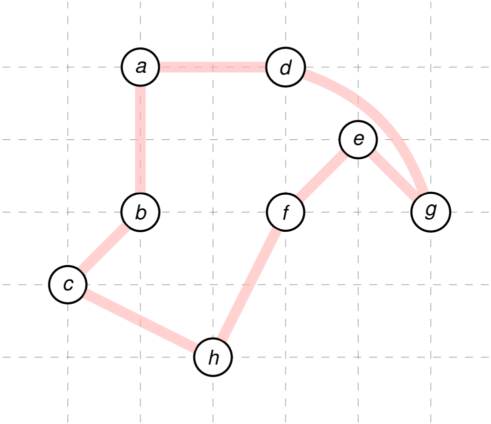


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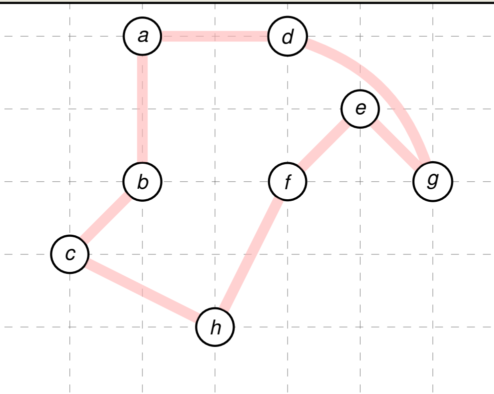


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