V. Approximation Algorithms via Exact Algorithms

Thomas Sauerwald

Easter 2017

UNIVERSITY OF CAMBRIDGE

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

The Subset-Sum Problem

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

The Subset-Sum Problem

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

This problem is NP-hard

The Subset-Sum Problem

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

$t = 13$ tons

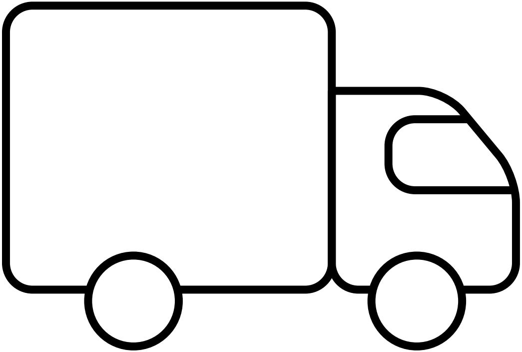
$x_1 = 10$

$x_2 = 4$

$x_3 = 5$

$x_4 = 6$

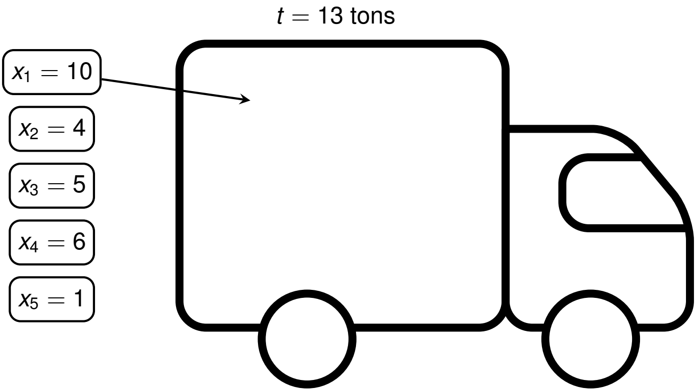
$x_5 = 1$



The Subset-Sum Problem

The Subset-Sum Problem

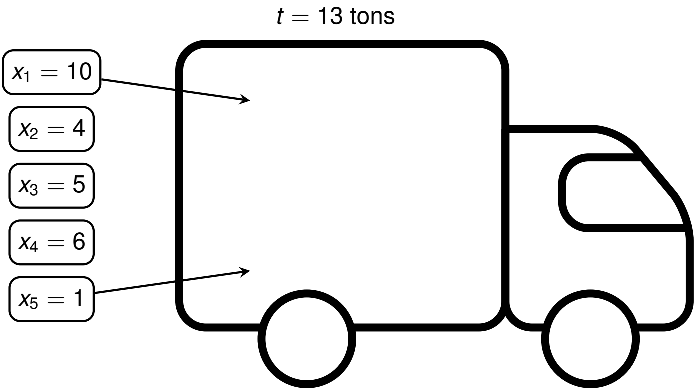
- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\, x_i \in S'} x_i \leq t$.



The Subset-Sum Problem

The Subset-Sum Problem

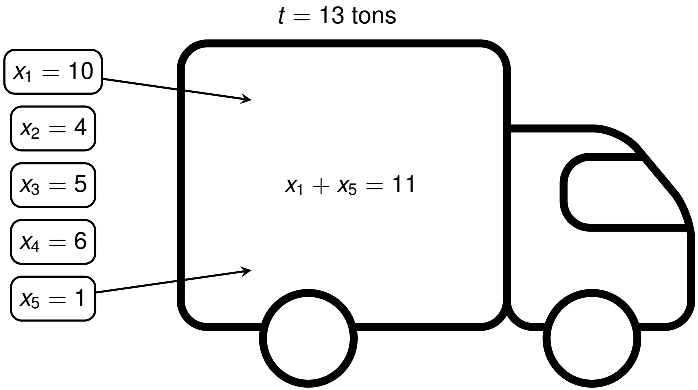
- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.



The Subset-Sum Problem

The Subset-Sum Problem

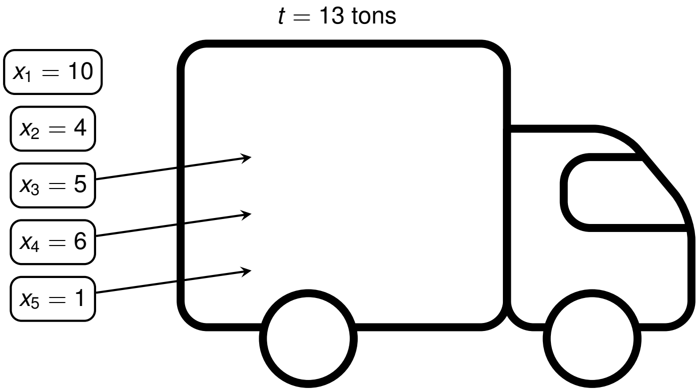
- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.



The Subset-Sum Problem

The Subset-Sum Problem

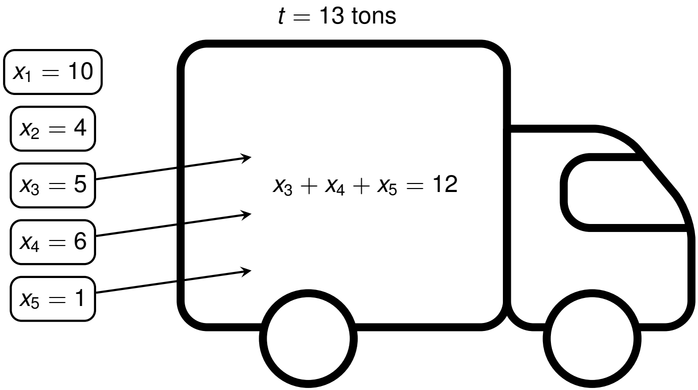
- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.



The Subset-Sum Problem

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\, x_i \in S'} x_i \leq t$.



An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

```
EXACT-SUBSET-SUM(S, t)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

$S + x := \{s + x : s \in S\}$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

```
EXACT-SUBSET-SUM(S, t)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

Returns the merged list (in sorted order and without duplicates)

$S + x := \{s + x : s \in S\}$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

implementable in time $O(|L_{i-1}|)$ (like Merge-Sort)

EXACT-SUBSET-SUM(S, t)

Returns the merged list (in sorted order and without duplicates)

1 $n = |S|$
2 $L_0 = \langle 0 \rangle$
3 **for** $i = 1$ **to** n
4 $\quad L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$ $\boxed{S + x := \{s + x \colon s \in S\}}$
5 $\quad$ remove from L_i every element that is greater than t
6 **return** the largest element in L_n

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

Example:

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

```
EXACT-SUBSET-SUM(S, t)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

Example:

- $S = \{1, 4, 5\}$, $t = 10$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

$\textsc{Exact-Subset-Sum}(S, t)$

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \textsc{Merge-Lists}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

```
EXACT-SUBSET-SUM(S, t)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
      remove from L_i every element that is greater than t
return the largest element in L_n
```

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$
- $L_3 = \langle 0, 1, 4, 5, 6, 9, 10 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

```
EXACT-SUBSET-SUM(S, t)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$
- $L_3 = \langle 0, 1, 4, 5, 6, 9, 10 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- **Correctness:** L_n contains all sums of $\{x_1, x_2, \ldots, x_n\}$

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$
- $L_3 = \langle 0, 1, 4, 5, 6, 9, 10 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

can be shown by induction on n

- **Correctness:** L_n contains all sums of $\{x_1, x_2, \ldots, x_n\}$

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$
- $L_3 = \langle 0, 1, 4, 5, 6, 9, 10 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- **Correctness:** L_n contains all sums of $\{x_1, x_2, \ldots, x_n\}$
- **Runtime:** $O(2^1 + 2^2 + \cdots + 2^n) = O(2^n)$

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$
- $L_3 = \langle 0, 1, 4, 5, 6, 9, 10 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- **Correctness:** L_n contains all sums of $\{x_1, x_2, \ldots, x_n\}$
- **Runtime:** $O(2^1 + 2^2 + \cdots + 2^n) = O(2^n)$

There are 2^i subsets of $\{x_1, x_2, \ldots, x_i\}$.

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$
- $L_3 = \langle 0, 1, 4, 5, 6, 9, 10 \rangle$

An Exact (Exponential-Time) Algorithm

Dynamic Progamming: Compute bottom-up all possible sums $\leq t$

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- **Correctness:** L_n contains all sums of $\{x_1, x_2, \ldots, x_n\}$
- **Runtime:** $O(2^1 + 2^2 + \cdots + 2^n) = O(2^n)$

There are 2^i subsets of $\{x_1, x_2, \ldots, x_i\}$.

Better runtime if t and/or $|L_i|$ are small.

Example:

- $S = \{1, 4, 5\}$, $t = 10$
- $L_0 = \langle 0 \rangle$
- $L_1 = \langle 0, 1 \rangle$
- $L_2 = \langle 0, 1, 4, 5 \rangle$
- $L_3 = \langle 0, 1, 4, 5, 6, 9, 10 \rangle$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

- $L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

- $L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$
- $\delta = 0.1$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

- $L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$
- $\delta = 0.1$
- $L' = \langle 10, 12, 15, 20, 23, 29 \rangle$

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

Towards a FPTAS

Idea: Don't need to maintain two values in L which are close to each other.

Trimming a List

- Given a trimming parameter $0 < \delta < 1$
- Trimming L yields minimal sublist L' so that for every $y \in L$: $\exists z \in L'$:

$$\frac{y}{1+\delta} \leq z \leq y.$$

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

TRIM works in time $\Theta(m)$, if L is given in sorted order.

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)      // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

Illustration of the Trim Operation

TRIM(L, δ)
```
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10 \rangle$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12 \rangle$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12 \rangle$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15 \rangle$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$

i

$L' = \langle 10, 12, 15 \rangle$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, \overset{\text{last}}{\overset{\downarrow}{15}}, \underset{\underset{\text{i}}{}}{\underset{\uparrow}{20}}, 21, 22, 23, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

```
let m be the length of L
L' = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L'
        last = y_i
return L'
```

$$\delta = 0.1$$

$$L = \langle 10, 11, 12, 15, \overset{\text{last}}{\overset{\downarrow}{20}}, 21, 22, \underset{\text{i}}{\underset{\uparrow}{23}}, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

$$L = \langle 10, 11, 12, 15, \overset{\text{last}\,\downarrow}{20}, 21, 22, \underset{\uparrow\,\text{i}}{23}, 24, 29 \rangle$$

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$\delta = 0.1$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23 \rangle$$

Illustration of the Trim Operation

TRIM(L, δ)

let m be the length of L
$L' = \langle y_1 \rangle$
$last = y_1$
for $i = 2$ **to** m
  if $y_i > last \cdot (1 + \delta)$ // $y_i \geq last$ because L is sorted
    append y_i onto the end of L'
    $last = y_i$
return L'

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$$

Illustration of the Trim Operation

```
TRIM(L, δ)
let m be the length of L
L′ = ⟨y_1⟩
last = y_1
for i = 2 to m
    if y_i > last · (1 + δ)        // y_i ≥ last because L is sorted
        append y_i onto the end of L′
        last = y_i
return L′
```

$$\delta = 0.1$$

last

$$L = \langle 10, 11, 12, 15, 20, 21, 22, 23, 24, 29 \rangle$$

i

$$L' = \langle 10, 12, 15, 20, 23, 29 \rangle$$

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = $ MERGE-LISTS$(L_{i-1}, L_{i-1} + x_i)$
    $L_i = $ TRIM$(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

The FPTAS

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
      L_i = TRIM(L_i, ε/2n)
      remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

Repeated application of TRIM to make sure L_i's remain short.

```
EXACT-SUBSET-SUM(S, t)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
      remove from L_i every element that is greater than t
return the largest element in L_n
```

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Repeated application of TRIM to make sure L_i's remain short.

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- We must bound the inaccuracy introduced by repeated trimming

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $\boxed{L_i = \text{TRIM}(L_i, \epsilon/2n)}$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

Repeated application of TRIM to make sure L_i's remain short.

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

The FPTAS

APPROX-SUBSET-SUM(S, t, ϵ)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

Repeated application of TRIM to make sure L_i's remain short.

EXACT-SUBSET-SUM(S, t)

```
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    remove from L_i every element that is greater than t
return the largest element in L_n
```

- We must bound the inaccuracy introduced by repeated trimming
- We must show that the algorithm is polynomial time

Solution is a careful choice of δ!

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ϵ)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ϵ/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
n = |S|
L_0 = ⟨0⟩
for i = 1 to n
    L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
    L_i = TRIM(L_i, ε/2n)
    remove from L_i every element that is greater than t
let z* be the largest value in L_n
return z*
```

- Input: $S = \langle 104, 102, 201, 101\rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ϵ/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

Running through an Example

```
Approx-Subset-Sum(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = Merge-Lists(L_{i-1}, L_{i-1} + x_i)
5      L_i = Trim(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ϵ)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ϵ/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

1 $n = |S|$
2 $L_0 = \langle 0 \rangle$
3 **for** $i = 1$ **to** n
4 $\quad L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
5 $\quad L_i = \text{TRIM}(L_i, \epsilon/2n)$
6 $\quad$ remove from L_i every element that is greater than t
7 let z^* be the largest value in L_n
8 **return** z^*

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i−1}, L_{i−1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

Running through an Example

APPROX-SUBSET-SUM(S, t, ϵ)

```
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ϵ)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ϵ/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$
- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$
- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$
- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$
- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$
- line 6: $L_4 = \langle 0, 101, 201, 302 \rangle$

Running through an Example

```
APPROX-SUBSET-SUM(S, t, ε)
1  n = |S|
2  L_0 = ⟨0⟩
3  for i = 1 to n
4      L_i = MERGE-LISTS(L_{i-1}, L_{i-1} + x_i)
5      L_i = TRIM(L_i, ε/2n)
6      remove from L_i every element that is greater than t
7  let z* be the largest value in L_n
8  return z*
```

- Input: $S = \langle 104, 102, 201, 101 \rangle$, $t = 308$, $\epsilon = 0.4$

$\Rightarrow$ Trimming parameter: $\delta = \epsilon/(2 \cdot n) = \epsilon/8 = 0.05$

- line 2: $L_0 = \langle 0 \rangle$

- line 4: $L_1 = \langle 0, 104 \rangle$
- line 5: $L_1 = \langle 0, 104 \rangle$
- line 6: $L_1 = \langle 0, 104 \rangle$

- line 4: $L_2 = \langle 0, 102, 104, 206 \rangle$
- line 5: $L_2 = \langle 0, 102, 206 \rangle$
- line 6: $L_2 = \langle 0, 102, 206 \rangle$

- line 4: $L_3 = \langle 0, 102, 201, 206, 303, 407 \rangle$
- line 5: $L_3 = \langle 0, 102, 201, 303, 407 \rangle$
- line 6: $L_3 = \langle 0, 102, 201, 303 \rangle$

- line 4: $L_4 = \langle 0, 101, 102, 201, 203, 302, 303, 404 \rangle$
- line 5: $L_4 = \langle 0, 101, 201, 302, 404 \rangle$
- line 6: $L_4 = \langle 0, 101, 201, 302 \rangle$

Returned solution $z^* = 302$, which is 2% within the optimum $307 = 104 + 102 + 101$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L'_i$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y$$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y$$

Can be shown by induction on i

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \stackrel{y=y^*, i=n}{\Rightarrow}$$

Can be shown by induction on i

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L'_i$ s.t.:

$$\frac{y}{(1 + \epsilon/(2n))^i} \leq z \leq y \quad \stackrel{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1 + \epsilon/(2n))^n} \leq z \leq y^*$$

Can be shown by induction on i

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \overset{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

$$\frac{y^*}{z} \leq \left(1 + \frac{\epsilon}{2n}\right)^n,$$

Can be shown by induction on i

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \overset{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

$$\frac{y^*}{z} \leq \left(1 + \frac{\epsilon}{2n}\right)^n,$$

Can be shown by induction on i

and now using the fact that $\left(1 + \frac{\epsilon/2}{n}\right)^n \overset{n\to\infty}{\longrightarrow} e^{\epsilon/2}$ yields

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \overset{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

Can be shown by induction on i

$$\frac{y^*}{z} \leq \left(1 + \frac{\epsilon}{2n}\right)^n,$$

and now using the fact that $\left(1 + \frac{\epsilon/2}{n}\right)^n \overset{n \to \infty}{\longrightarrow} e^{\epsilon/2}$ yields

$$\frac{y^*}{z} \leq e^{\epsilon/2}$$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \overset{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

Can be shown by induction on i

$$\frac{y^*}{z} \leq \left(1 + \frac{\epsilon}{2n}\right)^n,$$

and now using the fact that $\left(1 + \frac{\epsilon/2}{n}\right)^n \overset{n\to\infty}{\longrightarrow} e^{\epsilon/2}$ yields

$$\frac{y^*}{z} \leq e^{\epsilon/2}$$

Taylor approximation of e

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \stackrel{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

Can be shown by induction on i

$$\frac{y^*}{z} \leq \left(1+\frac{\epsilon}{2n}\right)^n,$$

and now using the fact that $\left(1+\frac{\epsilon/2}{n}\right)^n \stackrel{n\to\infty}{\longrightarrow} e^{\epsilon/2}$ yields

$$\frac{y^*}{z} \leq e^{\epsilon/2}$$

Taylor approximation of e

$$\leq 1+\epsilon/2+(\epsilon/2)^2$$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Approximation Ratio):

- Returned solution z^* is a valid solution ✓
- Let y^* denote an optimal solution
- For every possible sum $y \leq t$ of $x_1, \ldots, x_i$, there exists an element $z \in L_i'$ s.t.:

$$\frac{y}{(1+\epsilon/(2n))^i} \leq z \leq y \quad \stackrel{y=y^*, i=n}{\Rightarrow} \quad \frac{y^*}{(1+\epsilon/(2n))^n} \leq z \leq y^*$$

Can be shown by induction on i

$$\frac{y^*}{z} \leq \left(1+\frac{\epsilon}{2n}\right)^n,$$

and now using the fact that $\left(1+\frac{\epsilon/2}{n}\right)^n \stackrel{n\to\infty}{\longrightarrow} e^{\epsilon/2}$ yields

$$\frac{y^*}{z} \leq e^{\epsilon/2}$$

Taylor approximation of e

$$\leq 1+\epsilon/2+(\epsilon/2)^2 \leq 1+\epsilon$$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

Analysis of Approx-Subset-Sum

Theorem 35.8

Approx-Subset-Sum is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values.

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values. Hence,

$$\log_{1+\epsilon/(2n)} t + 2 =$$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values. Hence,

$$\log_{1+\epsilon/(2n)} t + 2 = \frac{\ln t}{\ln(1 + \epsilon/(2n))} + 2$$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values. Hence,

$$\log_{1+\epsilon/(2n)} t + 2 = \frac{\ln t}{\ln(1 + \epsilon/(2n))} + 2$$

For $x > -1$, $\ln(1 + x) \geq \frac{x}{1+x}$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values. Hence,

$$\log_{1+\epsilon/(2n)} t + 2 = \frac{\ln t}{\ln(1 + \epsilon/(2n))} + 2$$

$$\leq \frac{2n(1 + \epsilon/(2n)) \ln t}{\epsilon} + 2$$

For $x > -1$, $\ln(1 + x) \geq \frac{x}{1+x}$

Analysis of Approx-Subset-Sum

Theorem 35.8

Approx-Subset-Sum is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values. Hence,

$$\begin{aligned}
\log_{1+\epsilon/(2n)} t + 2 &= \frac{\ln t}{\ln(1 + \epsilon/(2n))} + 2 \\
&\leq \frac{2n(1 + \epsilon/(2n)) \ln t}{\epsilon} + 2 \\
&< \frac{3n \ln t}{\epsilon} + 2.
\end{aligned}$$

For $x > -1$, $\ln(1+x) \geq \frac{x}{1+x}$

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values. Hence,

$$\begin{aligned}\log_{1+\epsilon/(2n)} t + 2 &= \frac{\ln t}{\ln(1 + \epsilon/(2n))} + 2 \\ &\leq \frac{2n(1 + \epsilon/(2n)) \ln t}{\epsilon} + 2 \\ &< \frac{3n \ln t}{\epsilon} + 2.\end{aligned}$$

For $x > -1$, $\ln(1 + x) \geq \frac{x}{1+x}$

- This bound on $|L_i|$ is polynomial in the size of the input and in $1/\epsilon$.

Analysis of APPROX-SUBSET-SUM

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Proof (Running Time):

- Strategy: Derive a bound on $|L_i|$ (running time is linear in $|L_i|$)
- After trimming, two successive elements z and z' satisfy $z'/z \geq 1 + \epsilon/(2n)$

$\Rightarrow$ Possible Values after trimming are 0, 1, and up to $\lfloor \log_{1+\epsilon/(2n)} t \rfloor$ additional values. Hence,

$$\begin{aligned}\log_{1+\epsilon/(2n)} t + 2 &= \frac{\ln t}{\ln(1+\epsilon/(2n))} + 2 \\ &\leq \frac{2n(1+\epsilon/(2n))\ln t}{\epsilon} + 2 \\ &< \frac{3n \ln t}{\epsilon} + 2.\end{aligned}$$

For $x > -1$, $\ln(1+x) \geq \frac{x}{1+x}$

- This bound on $|L_i|$ is polynomial in the size of the input and in $1/\epsilon$. □

Need $\log(t)$ bits to represent t and n bits to represent S

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i\colon x_i \in S'} x_i \leq t$.

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\, x_i \in S'} x_i \leq t$.

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

The Knapsack Problem

- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\, x_i \in S'} x_i \leq t$.

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

The Knapsack Problem

- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t
- Goal: Find a subset $S' \subseteq S$ which

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

The Knapsack Problem

- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t
- Goal: Find a subset $S' \subseteq S$ which
  1. maximizes $\sum_{i \in S'} v_i$
  2. satisfies $\sum_{i \in S'} w_i \leq t$

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\ x_i \in S'} x_i \leq t$.

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

A more general problem than Subset-Sum

The Knapsack Problem

- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t
- Goal: Find a subset $S' \subseteq S$ which
  1. maximizes $\sum_{i \in S'} v_i$
  2. satisfies $\sum_{i \in S'} w_i \leq t$

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\, x_i \in S'} x_i \leq t$.

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

A more general problem than Subset-Sum

The Knapsack Problem

- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t
- Goal: Find a subset $S' \subseteq S$ which
  1. maximizes $\sum_{i \in S'} v_i$
  2. satisfies $\sum_{i \in S'} w_i \leq t$

Theorem

There is a FPTAS for the Knapsack problem.

Concluding Remarks

The Subset-Sum Problem

- Given: Set of positive integers $S = \{x_1, x_2, \ldots, x_n\}$ and positive integer t
- Goal: Find a subset $S' \subseteq S$ which maximizes $\sum_{i:\, x_i \in S'} x_i \leq t$.

Theorem 35.8

APPROX-SUBSET-SUM is a FPTAS for the subset-sum problem.

A more general problem than Subset-Sum

The Knapsack Problem

- Given: Items $i = 1, 2, \ldots, n$ with weights w_i and values v_i, and integer t
- Goal: Find a subset $S' \subseteq S$ which
  1. maximizes $\sum_{i \in S'} v_i$
  2. satisfies $\sum_{i \in S'} w_i \leq t$

Algorithm very similar to APPROX-SUBSET-SUM

Theorem

There is a FPTAS for the Knapsack problem.

Outline

The Subset-Sum Problem

Parallel Machine Scheduling

Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$

Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$
- Goal: Schedule the jobs on the machines minimizing the makespan $C_{\max} = \max_{1 \leq j \leq n} C_j$, where C_k is the completion time of job J_k.

Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$
- Goal: Schedule the jobs on the machines minimizing the makespan $C_{\max} = \max_{1 \leq j \leq n} C_j$, where C_k is the completion time of job J_k.

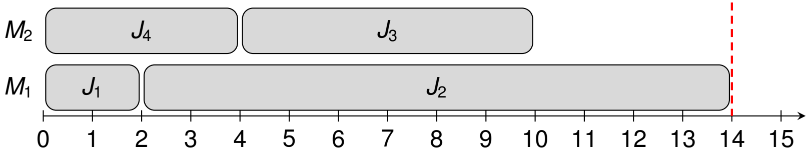
- J_1: $p_1 = 2$
- J_2: $p_2 = 12$
- J_3: $p_3 = 6$
- J_4: $p_4 = 4$

Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$
- Goal: Schedule the jobs on the machines minimizing the makespan $C_{\max} = \max_{1 \leq j \leq n} C_j$, where C_k is the completion time of job J_k.

- J_1: $p_1 = 2$
- J_2: $p_2 = 12$
- J_3: $p_3 = 6$
- J_4: $p_4 = 4$

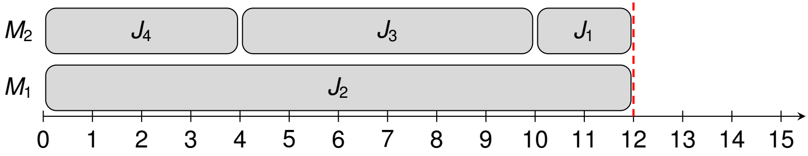


Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$
- Goal: Schedule the jobs on the machines minimizing the makespan $C_{\max} = \max_{1 \le j \le n} C_j$, where C_k is the completion time of job J_k.

- J_1: $p_1 = 2$
- J_2: $p_2 = 12$
- J_3: $p_3 = 6$
- J_4: $p_4 = 4$



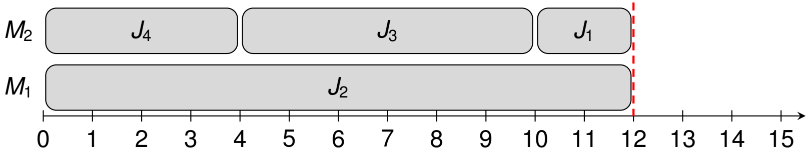
Parallel Machine Scheduling

Machine Scheduling Problem

- Given: n jobs $J_1, J_2, \ldots, J_n$ with processing times $p_1, p_2, \ldots, p_n$, and m identical machines $M_1, M_2, \ldots, M_m$
- Goal: Schedule the jobs on the machines minimizing the makespan $C_{\max} = \max_{1 \le j \le n} C_j$, where C_k is the completion time of job J_k.

- J_1: $p_1 = 2$
- J_2: $p_2 = 12$
- J_3: $p_3 = 6$
- J_4: $p_4 = 4$

For the analysis, it will be convenient to denote by C_i the completion time of a machine i.



NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

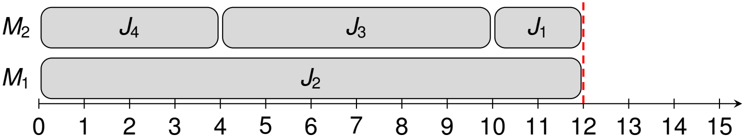
Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.

NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.

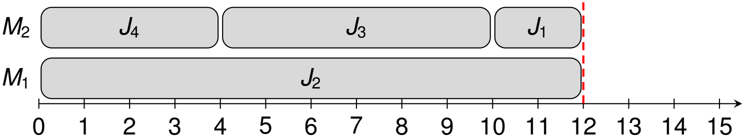


NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.



LIST SCHEDULING$(J_1, J_2, \ldots, J_n, m)$

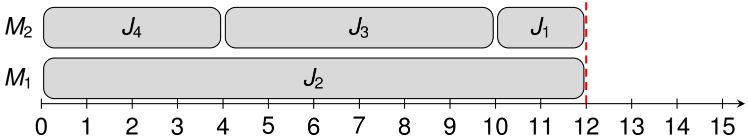
while there exists an unassigned job
    Schedule job on the machine with the least load

NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.



Equivalent to the following Online Algorithm [CLRS]:
Whenever a machine is idle, schedule any job that has not yet been scheduled.

LIST SCHEDULING$(J_1, J_2, \ldots, J_n, m)$

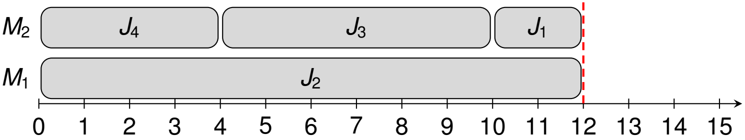
while there exists an unassigned job
    Schedule job on the machine with the least load

NP-Completeness of Parallel Machine Scheduling

Lemma

Parallel Machine Scheduling is NP-complete even if there are only two machines.

Proof Idea: Polynomial time reduction from NUMBER-PARTITIONING.



Equivalent to the following Online Algorithm [CLRS]:
Whenever a machine is idle, schedule any job that has not yet been scheduled.

LIST SCHEDULING($J_1, J_2, \ldots, J_n, m$)

while there exists an unassigned job
    Schedule job on the machine with the least load

How good is this most basic Greedy Approach?

List Scheduling Analysis (Observations)

List Scheduling Analysis (Observations)

Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C^*_{\max} \geq \max_{1 \leq k \leq n} p_k.$$

List Scheduling Analysis (Observations)

Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C^*_{\max} \geq \max_{1 \leq k \leq n} p_k.$$

b. The optimal makespan is at least as large as the average machine load, that is,

$$C^*_{\max} \geq \frac{1}{m} \sum_{k=1}^{n} p_k.$$

List Scheduling Analysis (Observations)

Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C^*_{\max} \geq \max_{1 \leq k \leq n} p_k.$$

b. The optimal makespan is at least as large as the average machine load, that is,

$$C^*_{\max} \geq \frac{1}{m} \sum_{k=1}^{n} p_k.$$

Proof:

List Scheduling Analysis (Observations)

Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C^*_{\max} \geq \max_{1 \leq k \leq n} p_k.$$

b. The optimal makespan is at least as large as the average machine load, that is,

$$C^*_{\max} \geq \frac{1}{m} \sum_{k=1}^{n} p_k.$$

Proof:

b. The total processing times of all n jobs equals $\sum_{k=1}^{n} p_k$

List Scheduling Analysis (Observations)

Ex 35-5 a.&b.

a. The optimal makespan is at least as large as the greatest processing time, that is,

$$C^*_{\max} \geq \max_{1 \leq k \leq n} p_k.$$

b. The optimal makespan is at least as large as the average machine load, that is,

$$C^*_{\max} \geq \frac{1}{m} \sum_{k=1}^{n} p_k.$$

Proof:

b. The total processing times of all n jobs equals $\sum_{k=1}^{n} p_k$

$\Rightarrow$ One machine must have a load of at least $\frac{1}{m} \cdot \sum_{k=1}^{n} p_k$ $\square$

List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$

List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

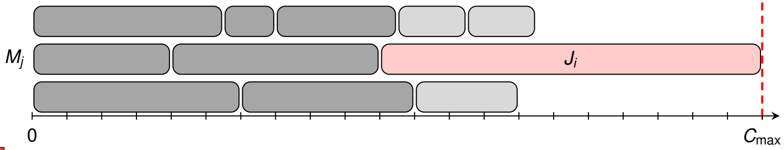
For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$



List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

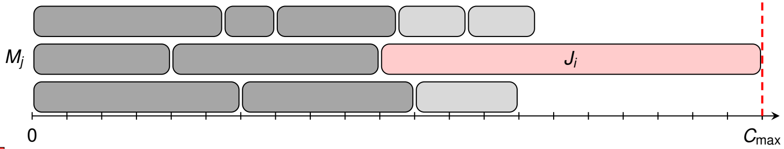
For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k .$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$
- When J_i was scheduled to machine M_j, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$



List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

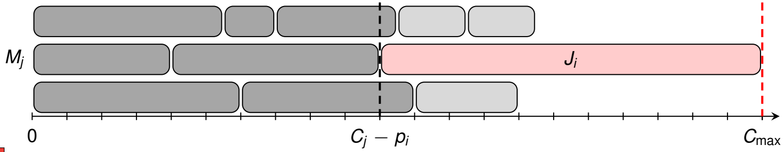
For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$
- When J_i was scheduled to machine M_j, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$



List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

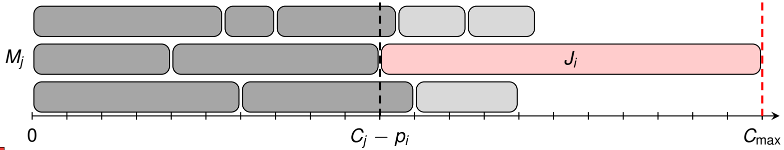
For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m}\sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$
- When J_i was scheduled to machine M_j, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$
- Averaging over k yields:



Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

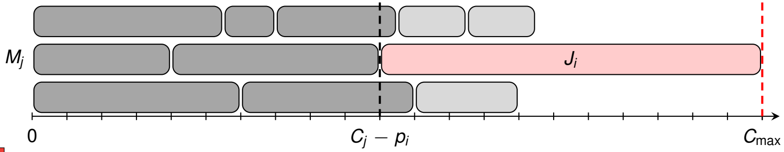
$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$
- When J_i was scheduled to machine M_j, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$
- Averaging over k yields:

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{m} C_k$$



List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

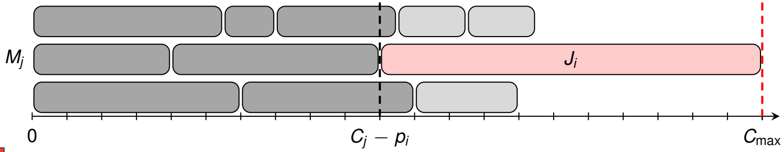
$$C_{\max} \leq \frac{1}{m} \sum_{k=1}^{n} p_k + \max_{1 \leq k \leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$
- When J_i was scheduled to machine M_j, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$
- Averaging over k yields:

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{m} C_k = \frac{1}{m} \sum_{k=1}^{n} p_k$$



List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

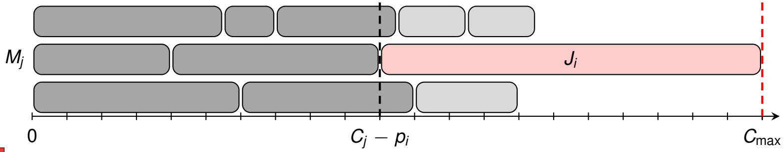
$$C_{\max} \leq \frac{1}{m}\sum_{k=1}^{n} p_k + \max_{1\leq k\leq n} p_k.$$

Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$
- When J_i was scheduled to machine M_j, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$
- Averaging over k yields:

$$C_j - p_i \leq \frac{1}{m}\sum_{k=1}^{m} C_k = \frac{1}{m}\sum_{k=1}^{n} p_k \quad \Rightarrow \quad C_j \leq \frac{1}{m}\sum_{k=1}^{n} p_k + \max_{1\leq k\leq n} p_k$$



List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m}\sum_{k=1}^{n} p_k + \max_{1\leq k\leq n} p_k.$$

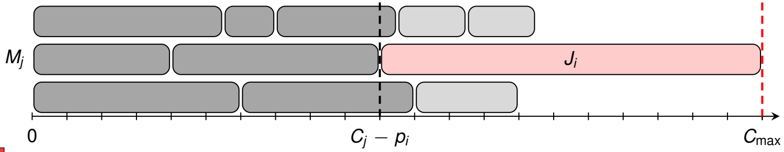
Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$
- When J_i was scheduled to machine M_j, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$
- Averaging over k yields:

Using Ex 35-5 a. & b.

$$C_j - p_i \leq \frac{1}{m}\sum_{k=1}^{m} C_k = \frac{1}{m}\sum_{k=1}^{n} p_k \quad \Rightarrow \quad C_j \leq \frac{1}{m}\sum_{k=1}^{n} p_k + \max_{1\leq k\leq n} p_k$$



List Scheduling Analysis (Final Step)

Ex 35-5 d. (Graham 1966)

For the schedule returned by the greedy algorithm it holds that

$$C_{\max} \leq \frac{1}{m}\sum_{k=1}^{n} p_k + \max_{1\leq k\leq n} p_k.$$

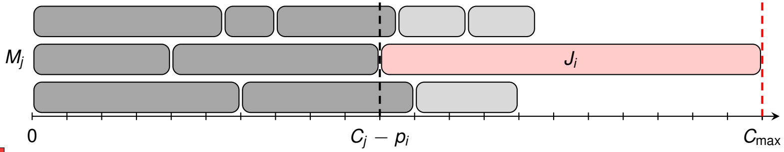
Hence list scheduling is a poly-time 2-approximation algorithm.

Proof:

- Let J_i be the last job scheduled on machine M_j with $C_{\max} = C_j$
- When J_i was scheduled to machine M_j, $C_j - p_i \leq C_k$ for all $1 \leq k \leq m$
- Averaging over k yields:

Using Ex 35-5 a. & b.

$$C_j - p_i \leq \frac{1}{m}\sum_{k=1}^{m} C_k = \frac{1}{m}\sum_{k=1}^{n} p_k \quad \Rightarrow \quad C_j \leq \frac{1}{m}\sum_{k=1}^{n} p_k + \max_{1\leq k\leq n} p_k \leq 2 \cdot C_{\max}^*$$



Improving Greedy

Analysis can be shown to be almost tight. Is there a better algorithm?

Improving Greedy

The problem of the List-Scheduling Approach were the large jobs

Analysis can be shown to be almost tight. Is there a better algorithm?

Improving Greedy

The problem of the List-Scheduling Approach were the large jobs

Analysis can be shown to be almost tight. Is there a better algorithm?

LEAST PROCESSING TIME($J_1, J_2, \ldots, J_n, m$)

```
Sort jobs decreasingly in their processing times
for i = 1 to m
    C_i = 0
    S_i = ∅
end for
for j = 1 to n
    i = argmin_{1≤k≤m} C_k
    S_i = S_i ∪ {j}, C_i = C_i + p_j
end for
return S_1, ..., S_m
```

Improving Greedy

The problem of the List-Scheduling Approach were the large jobs

Analysis can be shown to be almost tight. Is there a better algorithm?

LEAST PROCESSING TIME($J_1, J_2, \ldots, J_n, m$)

1: Sort jobs decreasingly in their processing times
2: **for** $i = 1$ to m
3: $\quad C_i = 0$
4: $\quad S_i = \emptyset$
5: **end for**
6: **for** $j = 1$ to n
7: $\quad i = \text{argmin}_{1 \leq k \leq m} C_k$
8: $\quad S_i = S_i \cup \{j\}$, $C_i = C_i + p_j$
9: **end for**
10: **return** $S_1, \ldots, S_m$

Runtime:

Improving Greedy

The problem of the List-Scheduling Approach were the large jobs

Analysis can be shown to be almost tight. Is there a better algorithm?

LEAST PROCESSING TIME($J_1, J_2, \ldots, J_n, m$)

Sort jobs decreasingly in their processing times
for $i = 1$ to m
    $C_i = 0$
    $S_i = \emptyset$
end for
for $j = 1$ to n
    $i = \text{argmin}_{1 \le k \le m} C_k$
    $S_i = S_i \cup \{j\}$, $C_i = C_i + p_j$
end for
return $S_1, \ldots, S_m$

Runtime:

- $O(n \log n)$ for sorting

Improving Greedy

The problem of the List-Scheduling Approach were the large jobs

Analysis can be shown to be almost tight. Is there a better algorithm?

LEAST PROCESSING TIME($J_1, J_2, \ldots, J_n, m$)

```
Sort jobs decreasingly in their processing times
for i = 1 to m
    C_i = 0
    S_i = ∅
end for
for j = 1 to n
    i = argmin_{1≤k≤m} C_k
    S_i = S_i ∪ {j}, C_i = C_i + p_j
end for
return S_1, ..., S_m
```

Runtime:

- $O(n \log n)$ for sorting
- $O(n \log m)$ for extracting (and re-inserting) the minimum (use priority queue).

Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

This can be shown to be tight (see next slide).

Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof (of approximation ratio $3/2$).

Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof (of approximation ratio $3/2$).

- Observation 1: If there are at most m jobs, then the solution is optimal.

Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof (of approximation ratio $3/2$).

- Observation 1: If there are at most m jobs, then the solution is optimal.
- Observation 2: If there are more than m jobs, then $C^*_{\max} \geq 2 \cdot p_{m+1}$.

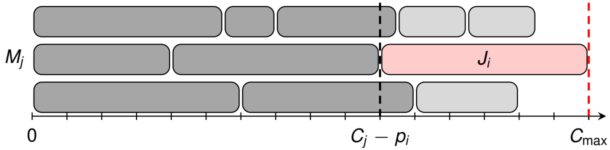
Analysis of Improved Greedy

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof (of approximation ratio 3/2).

- Observation 1: If there are at most m jobs, then the solution is optimal.
- Observation 2: If there are more than m jobs, then $C^*_{\max} \geq 2 \cdot p_{m+1}$.
- As in the analysis for list scheduling



Analysis of Improved Greedy

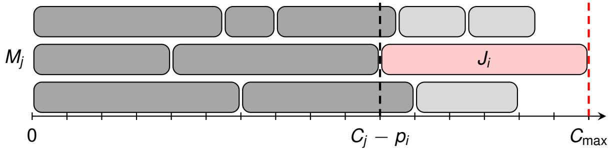
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof (of approximation ratio $3/2$).

- Observation 1: If there are at most m jobs, then the solution is optimal.
- Observation 2: If there are more than m jobs, then $C^*_{\max} \geq 2 \cdot p_{m+1}$.
- As in the analysis for list scheduling, we have

$$C_{\max} = C_j = (C_j - p_i) + p_i$$



Analysis of Improved Greedy

Graham 1966

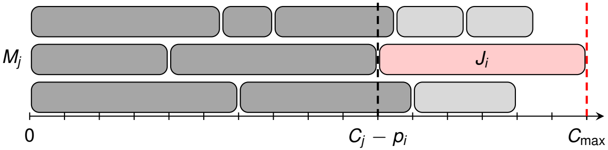
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof (of approximation ratio $3/2$).

- Observation 1: If there are at most m jobs, then the solution is optimal.
- Observation 2: If there are more than m jobs, then $C^*_{\max} \geq 2 \cdot p_{m+1}$.
- As in the analysis for list scheduling, we have

$$C_{\max} = C_j = (C_j - p_i) + p_i \leq C^*_{\max} + \frac{1}{2} C^*_{\max}$$

This is for the case $i \geq m+1$ (otherwise, an even stronger inequality holds)



Analysis of Improved Greedy

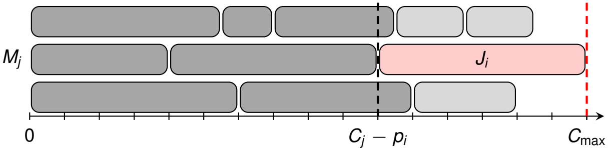
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof (of approximation ratio $3/2$).

- Observation 1: If there are at most m jobs, then the solution is optimal.
- Observation 2: If there are more than m jobs, then $C^*_{\max} \geq 2 \cdot p_{m+1}$.
- As in the analysis for list scheduling, we have

$$C_{\max} = C_j = (C_j - p_i) + p_i \leq C^*_{\max} + \frac{1}{2} C^*_{\max} = \frac{3}{2} C_{\max}. \qquad \square$$



Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines

Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

Tightness of the Bound for LPT

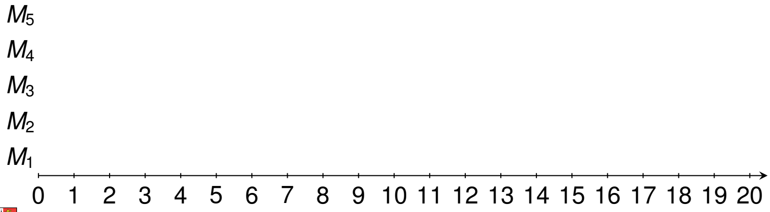
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



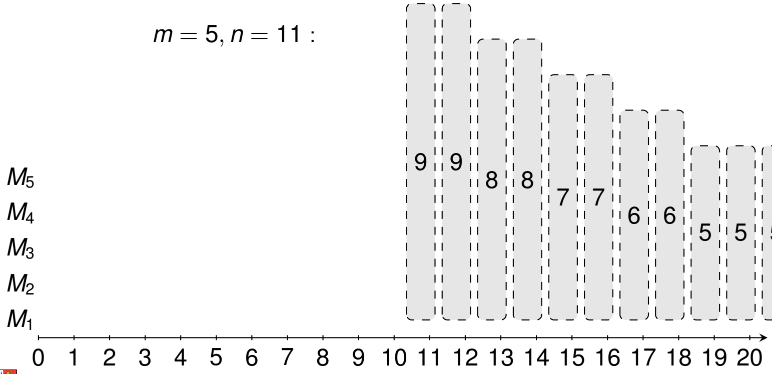
Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m



Tightness of the Bound for LPT

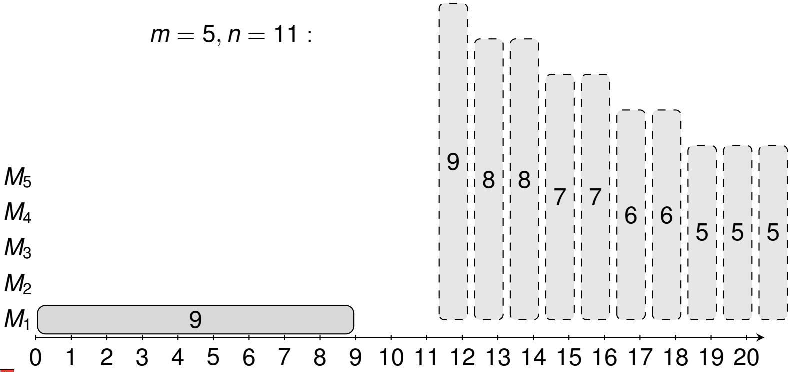
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

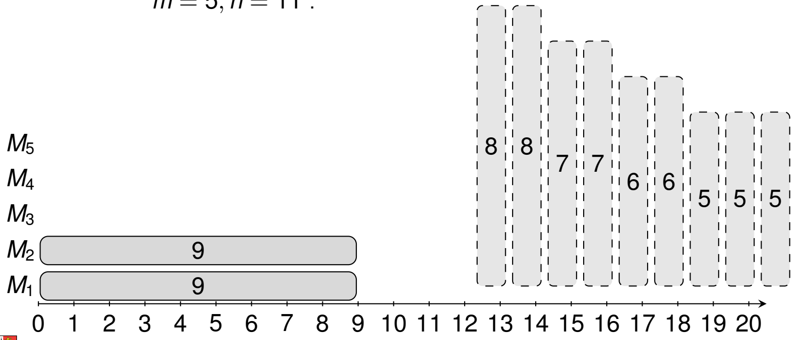
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

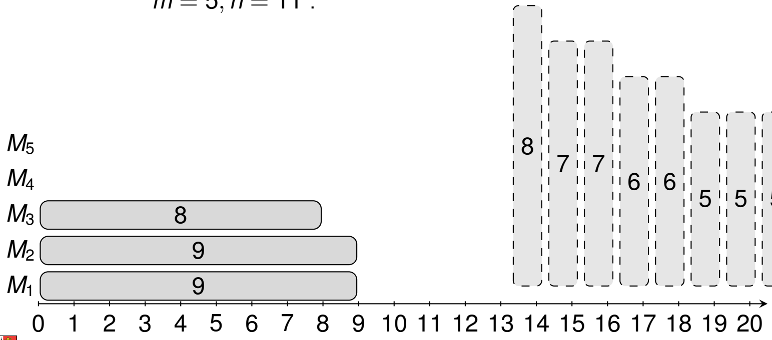
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m-1, 2m-2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

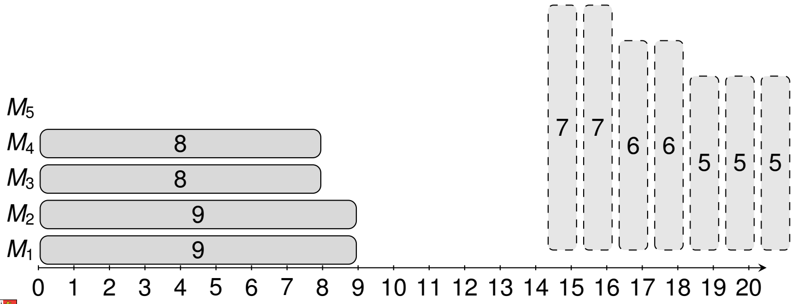
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

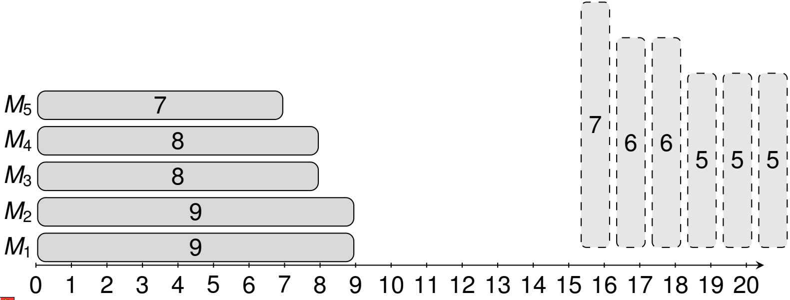
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

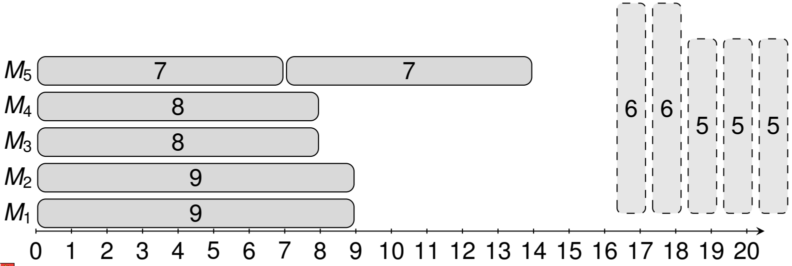
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

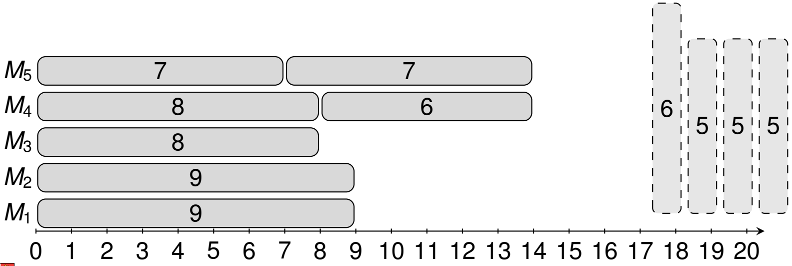
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

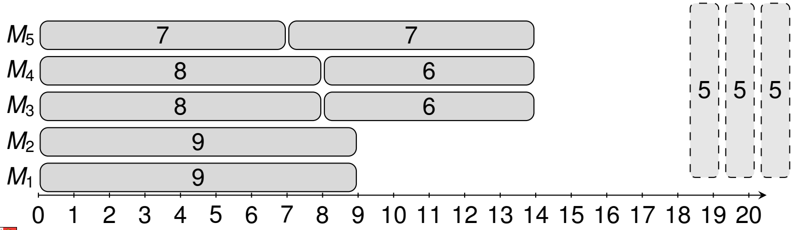
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

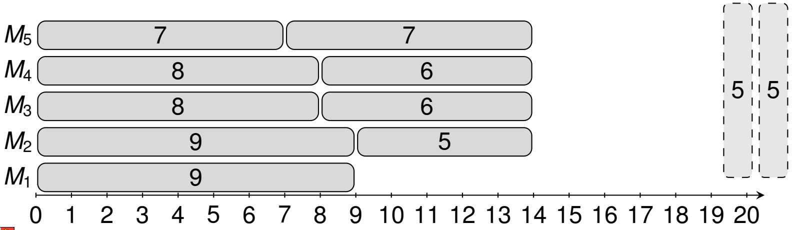
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

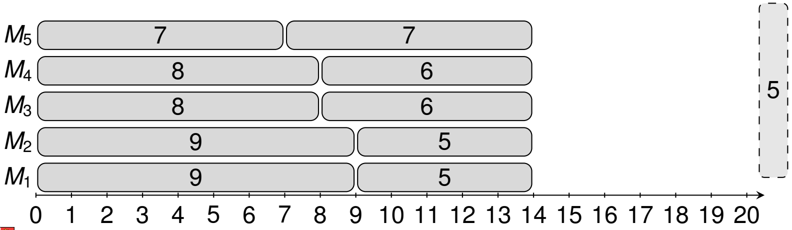
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m-1, 2m-2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



Tightness of the Bound for LPT

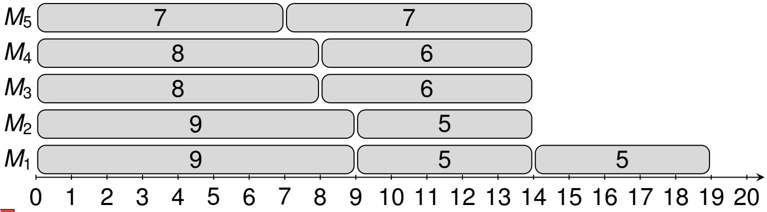
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \dots, m$ and one job of length m

$$m = 5, n = 11:$$



Tightness of the Bound for LPT

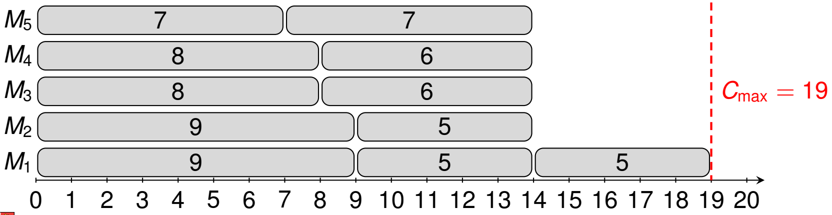
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:



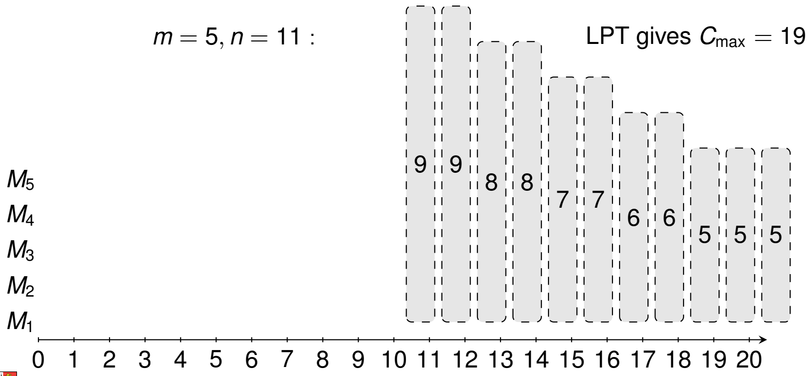
Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m



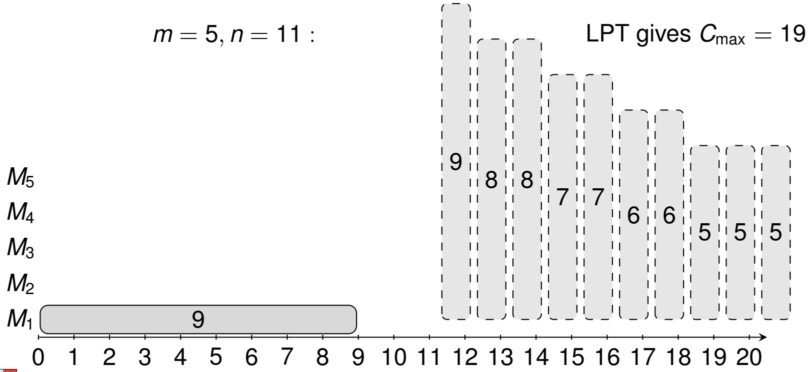
Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m



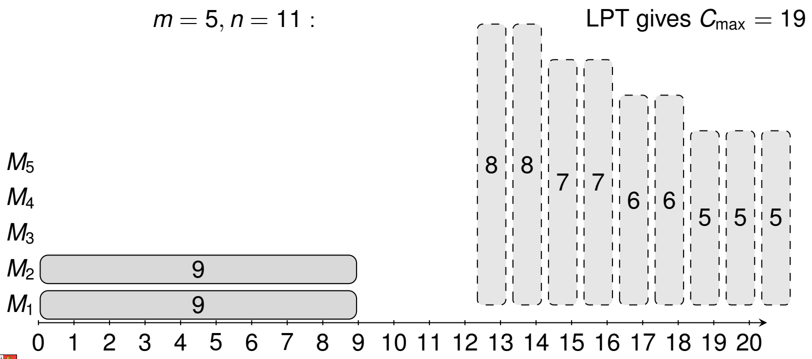
Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \dots, m$ and one job of length m



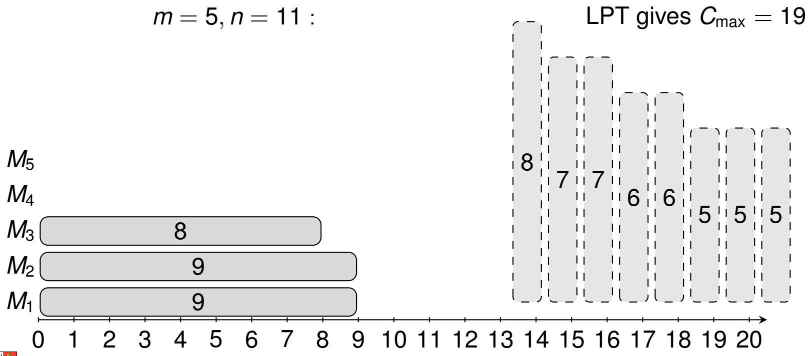
Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m-1, 2m-2, \ldots, m$ and one job of length m



Tightness of the Bound for LPT

Graham 1966

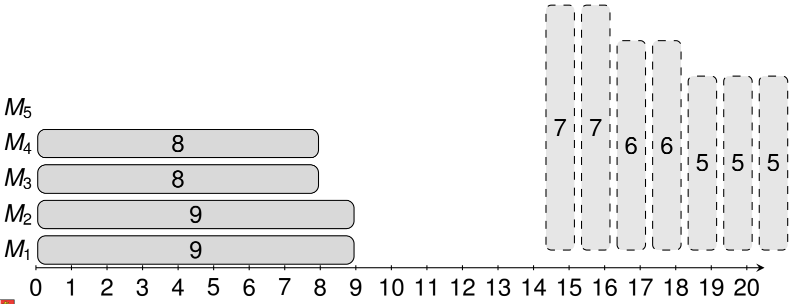
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$



Tightness of the Bound for LPT

Graham 1966

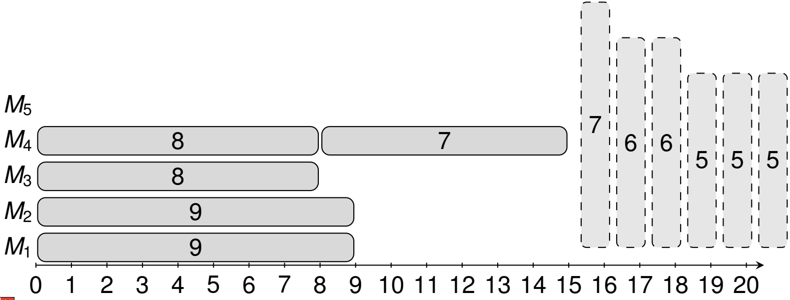
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$



Tightness of the Bound for LPT

Graham 1966

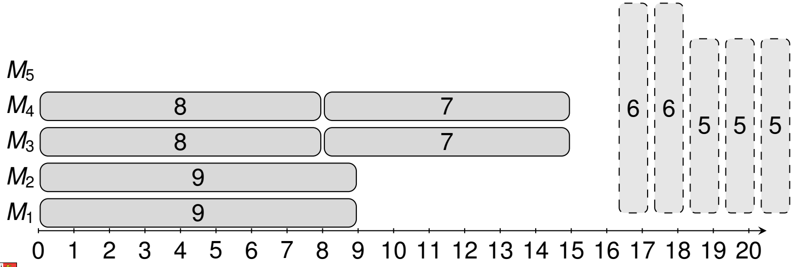
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$



Tightness of the Bound for LPT

Graham 1966

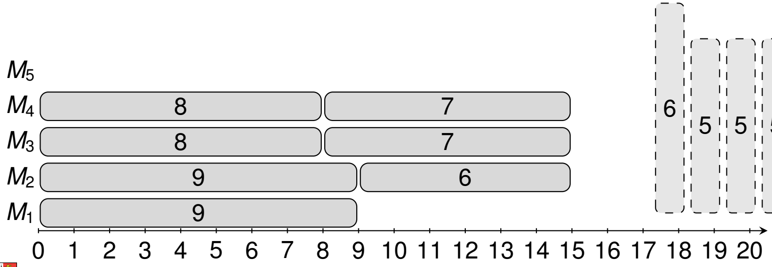
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m+1$ jobs of length $2m-1, 2m-2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$



Tightness of the Bound for LPT

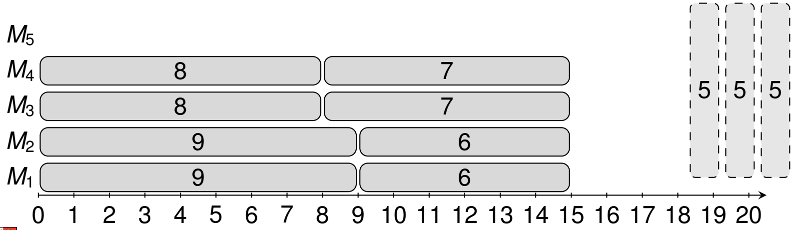
Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$: LPT gives $C_{\max} = 19$



Tightness of the Bound for LPT

Graham 1966

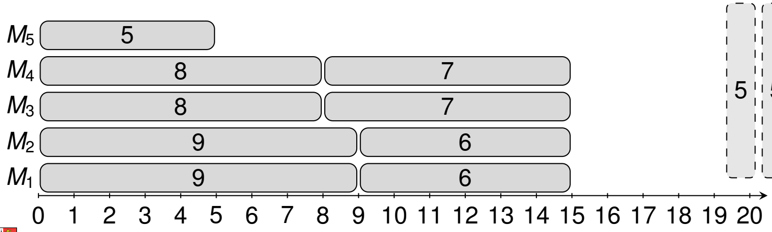
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$



Graham 1966

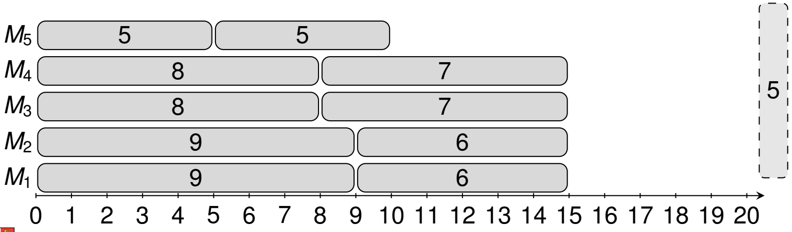
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$



Tightness of the Bound for LPT

Graham 1966

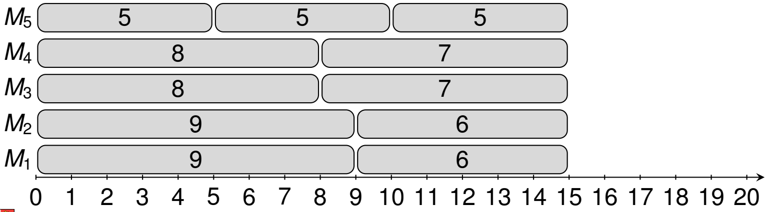
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$



Tightness of the Bound for LPT

Graham 1966

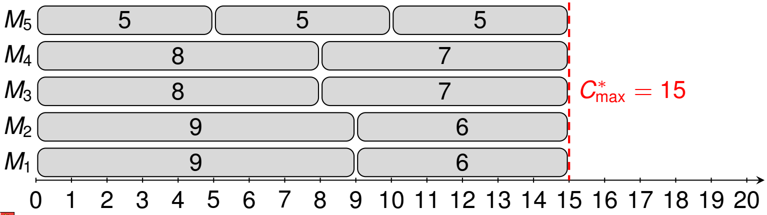
The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m-1, 2m-2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$



Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

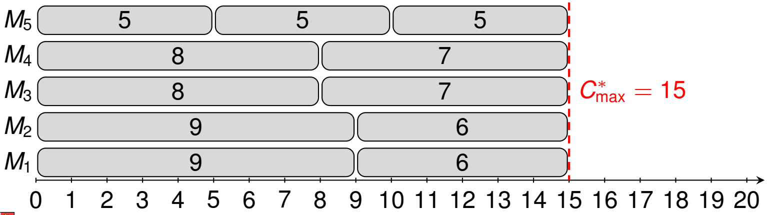
Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m-1, 2m-2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$

Optimum is $C^*_{\max} = 15$



Tightness of the Bound for LPT

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

$\frac{19}{15} = \frac{20}{15} - \frac{1}{15}$

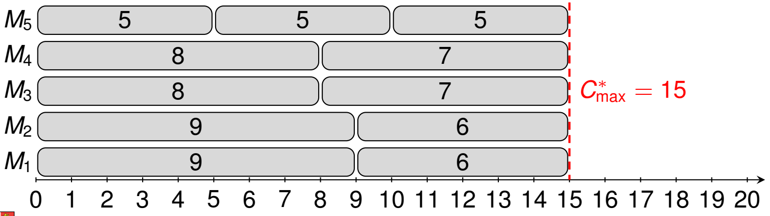
Proof of an instance which shows tightness:

- m machines
- $n = 2m + 1$ jobs of length $2m - 1, 2m - 2, \ldots, m$ and one job of length m

$m = 5, n = 11$:

LPT gives $C_{\max} = 19$

Optimum is $C^*_{\max} = 15$



A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Key Lemma

SUBROUTINE can be implemented in time $n^{O(1/\epsilon^2)}$.

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Key Lemma

We will prove this on the next slides.

SUBROUTINE can be implemented in time $n^{O(1/\epsilon^2)}$.

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

We will prove this on the next slides.

Key Lemma

SUBROUTINE can be implemented in time $n^{O(1/\epsilon^2)}$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

We will prove this on the next slides.

Key Lemma

SUBROUTINE can be implemented in time $n^{O(1/\epsilon^2)}$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Proof (using Key Lemma):

PTAS$(J_1, J_2, \ldots, J_n, m)$

1: Do binary search to find smallest T s.t. $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

2: **Return** solution computed by SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

$\textsc{Subroutine}(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$
2: Or: **Return** there is no solution with makespan $< T$

We will prove this on the next slides.

Key Lemma

$\textsc{Subroutine}$ can be implemented in time $n^{O(1/\epsilon^2)}$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Since $0 \leq C^*_{\max} \leq P$ and $C^*_{\max}$ is integral, binary search terminates after $O(\log P)$ steps.

Proof (using Key Lemma):

$\text{PTAS}(J_1, J_2, \ldots, J_n, m)$

1: Do binary search to find smallest T s.t. $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.
2: **Return** solution computed by $\textsc{Subroutine}(J_1, J_2, \ldots, J_n, m, T)$

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$
1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$
2: Or: **Return** there is no solution with makespan $< T$

We will prove this on the next slides.

Key Lemma

SUBROUTINE can be implemented in time $n^{O(1/\epsilon^2)}$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Since $0 \leq C^*_{\max} \leq P$ and $C^*_{\max}$ is integral, binary search terminates after $O(\log P)$ steps.

Proof (using Key Lemma):

PTAS$(J_1, J_2, \ldots, J_n, m)$
1: Do binary search to find smallest T s.t. $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.
2: **Return** solution computed by SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

A PTAS for Parallel Machine Scheduling

Basic Idea: For $(1+\epsilon)$-approximation, don't have to work with exact p_k's.

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$
1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$
2: Or: **Return** there is no solution with makespan $< T$

We will prove this on the next slides.

Key Lemma

SUBROUTINE can be implemented in time $n^{O(1/\epsilon^2)}$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

polynomial in the size of the input

Since $0 \leq C^*_{\max} \leq P$ and $C^*_{\max}$ is integral, binary search terminates after $O(\log P)$ steps.

Proof (using Key Lemma):

PTAS$(J_1, J_2, \ldots, J_n, m)$
1: Do binary search to find smallest T s.t. $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.
2: **Return** solution computed by SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

Implementation of Subroutine

$\textsc{Subroutine}(J_1, J_2, \ldots, J_n, m, T)$

Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

Or: **Return** there is no solution with makespan $< T$

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i : p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i : p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i \colon p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i : p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i \colon p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.
- Otherwise, let $i \in J_{\text{small}}$ be the last job added to M_j.

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$
2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i \colon p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.
- Otherwise, let $i \in J_{\text{small}}$ be the last job added to M_j.

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k$$

the "well-known" formula

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i \colon p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.
- Otherwise, let $i \in J_{\text{small}}$ be the last job added to M_j.

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k \qquad \Rightarrow$$

the "well-known" formula

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$
2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i \colon p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.
- Otherwise, let $i \in J_{\text{small}}$ be the last job added to M_j.

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k \qquad \Rightarrow \qquad C_j \leq p_i + \frac{1}{m} \sum_{k=1}^{n} p_k$$

the "well-known" formula

Implementation of Subroutine

$\textsc{Subroutine}(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i \colon p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.
- Otherwise, let $i \in J_{\text{small}}$ be the last job added to M_j.

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k \quad \Rightarrow \quad C_j \leq p_i + \frac{1}{m} \sum_{k=1}^{n} p_k$$

the "well-known" formula

$$\leq \epsilon \cdot T + C^*_{\max}$$

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i \colon p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.
- Otherwise, let $i \in J_{\text{small}}$ be the last job added to M_j.

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k \qquad \Rightarrow \qquad C_j \leq p_i + \frac{1}{m} \sum_{k=1}^{n} p_k$$

the "well-known" formula

$$\leq \epsilon \cdot T + C^*_{\max}$$

$$\leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\} \qquad \square$$

Implementation of Subroutine

SUBROUTINE$(J_1, J_2, \ldots, J_n, m, T)$

1: Either: **Return** a solution with $C_{\max} \leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\}$

2: Or: **Return** there is no solution with makespan $< T$

Observation

Divide jobs into two groups: $J_{\text{small}} = \{J_i : p_i \leq \epsilon \cdot T\}$ and $J_{\text{large}} = J \setminus J_{\text{small}}$. Given a solution for J_{large} only with makespan $(1+\epsilon) \cdot T$, then greedily placing J_{small} yields a solution with makespan $(1+\epsilon) \cdot \max\{T, C^*_{\max}\}$.

Proof:

- Let M_j be the machine with largest load
- If there are no jobs from J_{small}, then makespan is at most $(1+\epsilon) \cdot T$.
- Otherwise, let $i \in J_{\text{small}}$ be the last job added to M_j.

$$C_j - p_i \leq \frac{1}{m} \sum_{k=1}^{n} p_k \qquad \Rightarrow \qquad C_j \leq p_i + \frac{1}{m} \sum_{k=1}^{n} p_k$$

the "well-known" formula

$$\leq \epsilon \cdot T + C^*_{\max}$$

$$\leq (1+\epsilon) \cdot \max\{T, C^*_{\max}\} \qquad \square$$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$.

Proof of Key Lemma

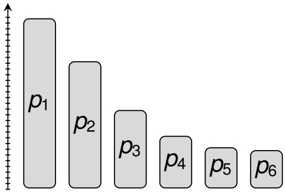
Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_j' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

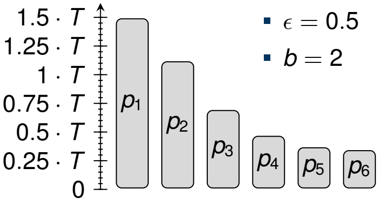
- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_j' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$



Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \le \epsilon$. Define processing times $p_i' = \left\lceil \frac{p_j b^2}{T} \right\rceil \cdot \frac{T}{b^2}$

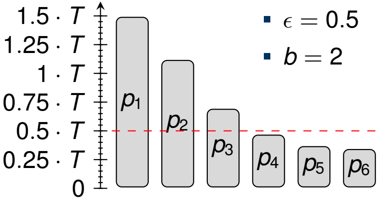


- $\epsilon = 0.5$
- $b = 2$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

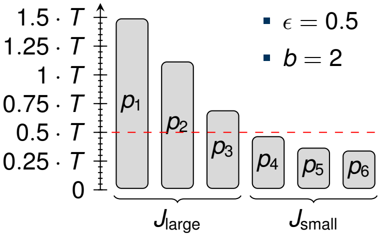


- $\epsilon = 0.5$
- $b = 2$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

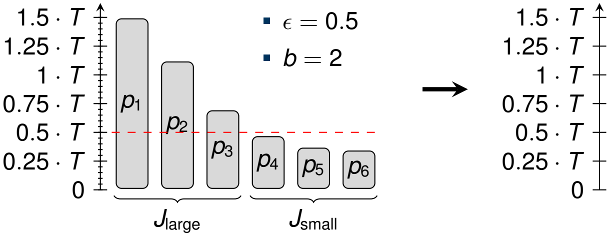


- $\epsilon = 0.5$
- $b = 2$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

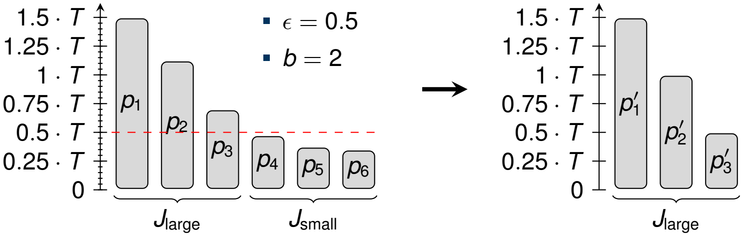
- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$



Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon)\cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

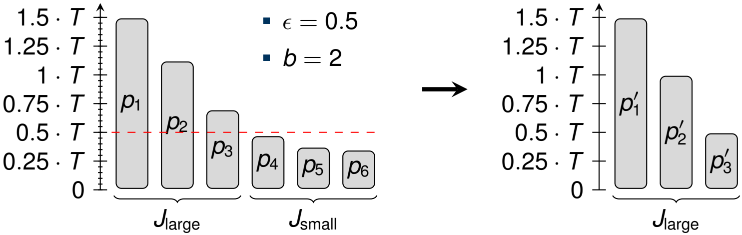


Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$ Can assume there are no jobs with $p_j \geq T$!



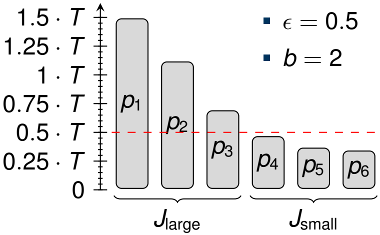
Proof of Key Lemma

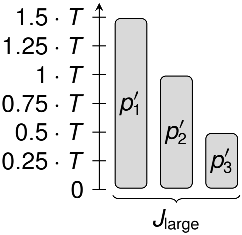
Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon)\cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.





Proof of Key Lemma

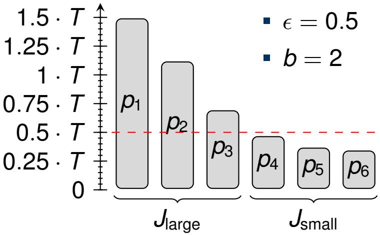
Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \left\lceil \frac{p_i b^2}{T} \right\rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

Assignments to one machine with makespan $\leq T$.



Proof of Key Lemma

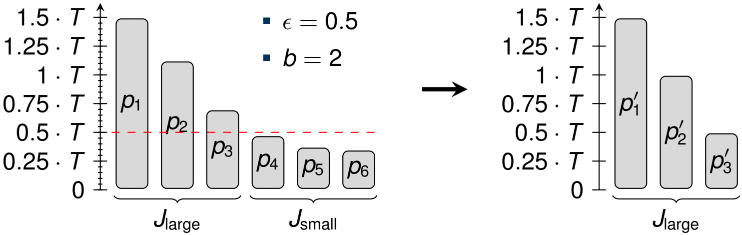
Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon)\cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:



Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon)\cdot T$.

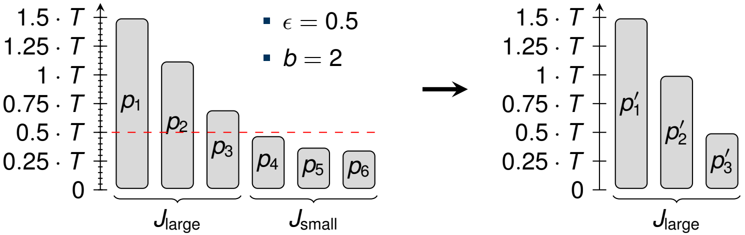
- Let b be the smallest integer with $1/b \le \epsilon$. Define processing times $p_i' = \lceil \frac{p_i b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \le T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\le T$:

$$f(0,0,\ldots,0) = 0$$



Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon)\cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \left\lceil \frac{p_i b^2}{T} \right\rceil \cdot \frac{T}{b^2}$

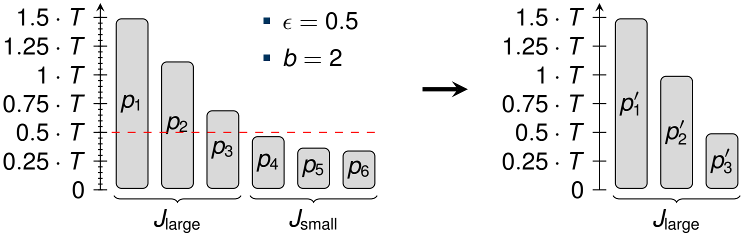
$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

$$f(0,0,\ldots,0) = 0$$

$$f(n_b, n_{b+1}, \ldots, n_{b^2}) = 1 + \min_{(s_b, s_{b+1}, \ldots, s_{b^2}) \in \mathcal{C}} f(n_b - s_b, n_{b+1} - s_{b+1}, \ldots, n_{b^2} - s_{b^2}).$$



Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon)\cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \left\lceil \frac{p_i b^2}{T} \right\rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

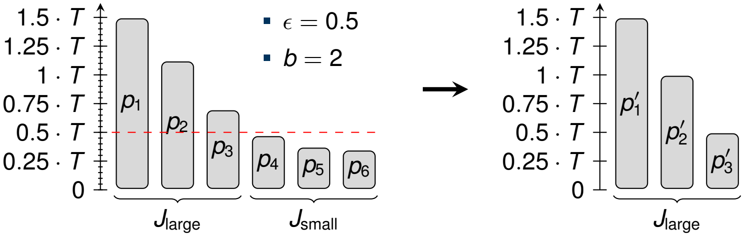
- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

Assign some jobs to one machine, and then use as few machines as possible for the rest.

$$f(0, 0, \ldots, 0) = 0$$

$$f(n_b, n_{b+1}, \ldots, n_{b^2}) = 1 + \min_{(s_b, s_{b+1}, \ldots, s_{b^2}) \in \mathcal{C}} f(n_b - s_b, n_{b+1} - s_{b+1}, \ldots, n_{b^2} - s_{b^2}).$$



Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

$$f(0,0,\ldots,0) = 0$$

$$f(n_b, n_{b+1}, \ldots, n_{b^2}) = 1 + \min_{(s_b, s_{b+1}, \ldots, s_{b^2}) \in \mathcal{C}} f(n_b - s_b, n_{b+1} - s_{b+1}, \ldots, n_{b^2} - s_{b^2}).$$

- Number of table entries is at most n^{b^2}, hence filling all entries takes $n^{O(b^2)}$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_i b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.
- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

$$f(0, 0, \ldots, 0) = 0$$

$$f(n_b, n_{b+1}, \ldots, n_{b^2}) = 1 + \min_{(s_b, s_{b+1}, \ldots, s_{b^2}) \in \mathcal{C}} f(n_b - s_b, n_{b+1} - s_{b+1}, \ldots, n_{b^2} - s_{b^2}).$$

- Number of table entries is at most n^{b^2}, hence filling all entries takes $n^{O(b^2)}$
- If $f(n_b, n_{b+1}, \ldots, n_{b^2}) \leq m$ (for the jobs with p'), then return yes, otherwise no.

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

$$f(0, 0, \ldots, 0) = 0$$

$$f(n_b, n_{b+1}, \ldots, n_{b^2}) = 1 + \min_{(s_b, s_{b+1}, \ldots, s_{b^2}) \in \mathcal{C}} f(n_b - s_b, n_{b+1} - s_{b+1}, \ldots, n_{b^2} - s_{b^2}).$$

- Number of table entries is at most n^{b^2}, hence filling all entries takes $n^{O(b^2)}$
- If $f(n_b, n_{b+1}, \ldots, n_{b^2}) \leq m$ (for the jobs with p'), then return yes, otherwise no.
- As every machine is assigned at most b jobs ($p_i' \geq \frac{T}{b}$) and the makespan is $\leq T$,

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon)\cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_j b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

$$f(0,0,\ldots,0) = 0$$

$$f(n_b, n_{b+1}, \ldots, n_{b^2}) = 1 + \min_{(s_b, s_{b+1}, \ldots, s_{b^2}) \in \mathcal{C}} f(n_b - s_b, n_{b+1} - s_{b+1}, \ldots, n_{b^2} - s_{b^2}).$$

- Number of table entries is at most n^{b^2}, hence filling all entries takes $n^{O(b^2)}$
- If $f(n_b, n_{b+1}, \ldots, n_{b^2}) \leq m$ (for the jobs with p'), then return yes, otherwise no.
- As every machine is assigned at most b jobs ($p_i' \geq \frac{T}{b}$) and the makespan is $\leq T$,

$$C_{\max} \leq T + b \cdot \max_{i \in J_{\text{large}}} (p_i - p_i')$$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \left\lceil \frac{p_i b^2}{T} \right\rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

$$f(0,0,\ldots,0) = 0$$

$$f(n_b, n_{b+1}, \ldots, n_{b^2}) = 1 + \min_{(s_b, s_{b+1}, \ldots, s_{b^2}) \in \mathcal{C}} f(n_b - s_b, n_{b+1} - s_{b+1}, \ldots, n_{b^2} - s_{b^2}).$$

- Number of table entries is at most n^{b^2}, hence filling all entries takes $n^{O(b^2)}$
- If $f(n_b, n_{b+1}, \ldots, n_{b^2}) \leq m$ (for the jobs with p'), then return yes, otherwise no.
- As every machine is assigned at most b jobs ($p_i' \geq \frac{T}{b}$) and the makespan is $\leq T$,

$$\begin{aligned} C_{\max} &\leq T + b \cdot \max_{i \in J_{\text{large}}} (p_i - p_i') \\ &\leq T + b \cdot \frac{T}{b^2} \end{aligned}$$

Proof of Key Lemma

Use Dynamic Programming to schedule J_{large} with makespan $(1+\epsilon) \cdot T$.

- Let b be the smallest integer with $1/b \leq \epsilon$. Define processing times $p_i' = \lceil \frac{p_i b^2}{T} \rceil \cdot \frac{T}{b^2}$

$\Rightarrow$ Every $p_i' = \alpha \cdot \frac{T}{b^2}$ for $\alpha = b, b+1, \ldots, b^2$

- Let $\mathcal{C}$ be all $(s_b, s_{b+1}, \ldots, s_{b^2})$ with $\sum_{i=j}^{b^2} s_j \cdot j \cdot \frac{T}{b^2} \leq T$.

- Let $f(n_b, n_{b+1}, \ldots, n_{b^2})$ be the minimum number of machines required to schedule all jobs with makespan $\leq T$:

$$f(0, 0, \ldots, 0) = 0$$

$$f(n_b, n_{b+1}, \ldots, n_{b^2}) = 1 + \min_{(s_b, s_{b+1}, \ldots, s_{b^2}) \in \mathcal{C}} f(n_b - s_b, n_{b+1} - s_{b+1}, \ldots, n_{b^2} - s_{b^2}).$$

- Number of table entries is at most n^{b^2}, hence filling all entries takes $n^{O(b^2)}$
- If $f(n_b, n_{b+1}, \ldots, n_{b^2}) \leq m$ (for the jobs with p'), then return yes, otherwise no.
- As every machine is assigned at most b jobs ($p_i' \geq \frac{T}{b}$) and the makespan is $\leq T$,

$$C_{\max} \leq T + b \cdot \max_{i \in J_{\text{large}}} (p_i - p_i')$$

$$\leq T + b \cdot \frac{T}{b^2} \leq (1+\epsilon) \cdot T. \qquad \square$$

Final Remarks

Graham 1966

List scheduling has an approximation ratio of 2.

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Final Remarks

Graham 1966

List scheduling has an approximation ratio of 2.

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Final Remarks

Graham 1966

List scheduling has an approximation ratio of 2.

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Can we find a FPTAS (for polynomially bounded processing times)?

Final Remarks

Graham 1966

List scheduling has an approximation ratio of 2.

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Can we find a FPTAS (for polynomially bounded processing times)? **No!**

Final Remarks

Graham 1966

List scheduling has an approximation ratio of 2.

Graham 1966

The LPT algorithm has an approximation ratio of $4/3 - 1/(3m)$.

Theorem (Hochbaum, Shmoys'87)

There exists a PTAS for Parallel Machine Scheduling which runs in time $O(n^{O(1/\epsilon^2)} \cdot \log P)$, where $P := \sum_{k=1}^{n} p_k$.

Can we find a FPTAS (for polynomially bounded processing times)?
No!

Because for sufficiently small approximation ratio $1 + \epsilon$, the computed solution has to be optimal, and Parallel Machine Scheduling is strongly NP-hard.