VII. Approximation Algorithms: Randomisation and Rounding

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Easter 2017

UNIVERSITY OF CAMBRIDGE

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

Performance Ratios for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost C of the returned solution and optimal cost C^* satisfy:

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Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

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Idea: What about assigning each variable independently at random?

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Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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Follows from the previous Corollary.

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$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

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One of the two conditional expectations is at least $\mathbf{E}[\,Y\,]$!

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$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

One of the two conditional expectations is at least $\mathbf{E}[\,Y\,]$!

Algorithm: Assign x_1 so that the conditional expectation is maximized and recurse.

Expected Approximation Ratio

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

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GREEDY-3-CNF(ϕ, n, m)

1: **for** $j = 1, 2, \ldots, n$

2: Compute $\mathbf{E}[\,Y \mid x_1 = v_1 \ldots, x_{j-1} = v_{j-1}, x_j = 1\,]$

3: Compute $\mathbf{E}[\,Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 0\,]$

4: Let $x_j = v_j$ so that the conditional expectation is maximized

5: **return** the assignment $v_1, v_2, \ldots, v_n$

Analysis of GREEDY-3-CNF(ϕ, n, m)

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

Analysis of GREEDY-3-CNF(ϕ, n, m)

This algorithm is deterministic.

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computable in $O(1)$

Analysis of Greedy-3-CNF(ϕ, n, m)

This algorithm is deterministic.

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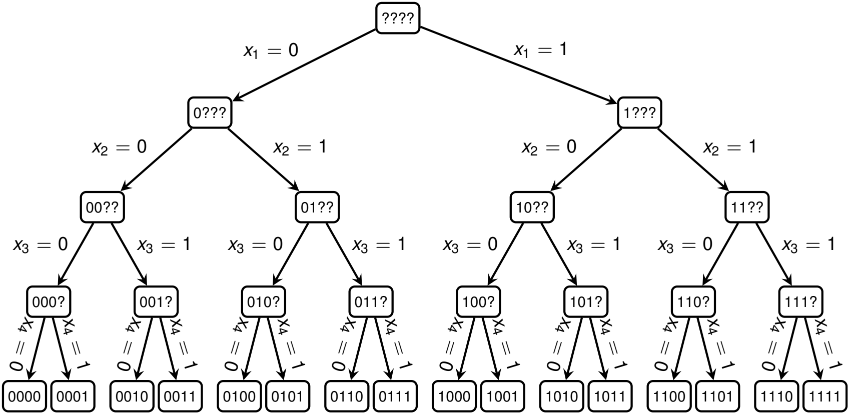
\end{aligned}$$

□

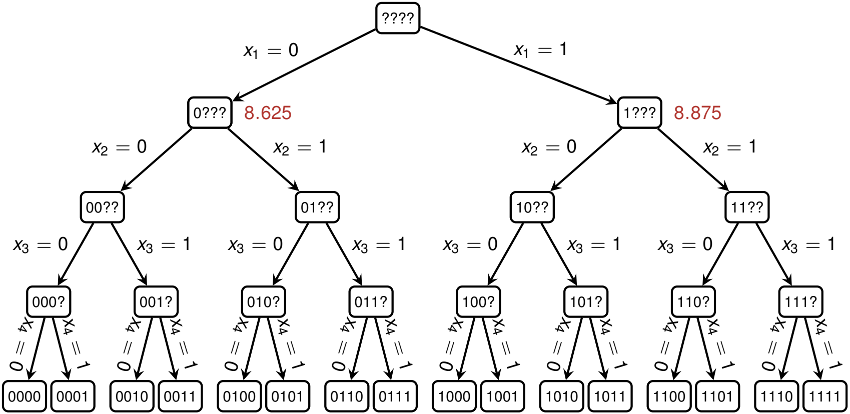
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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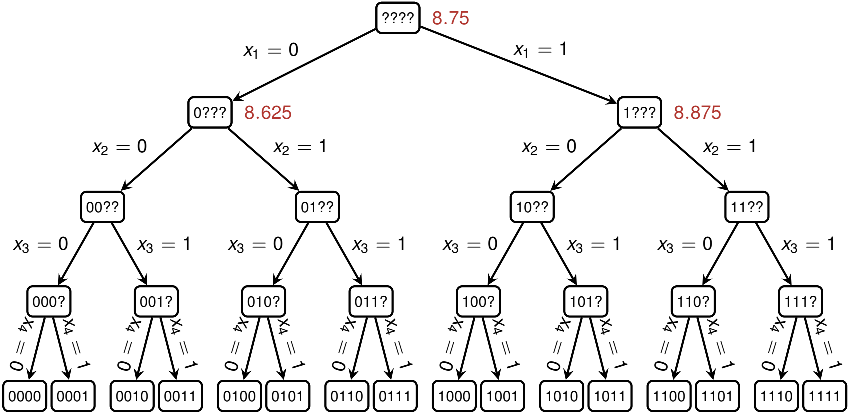
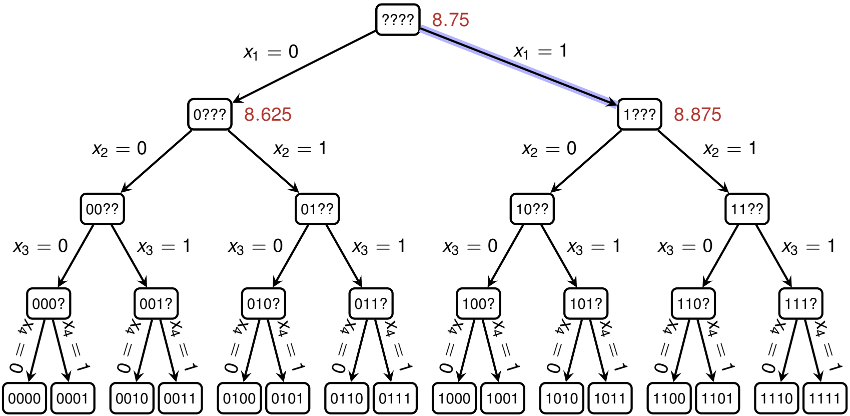


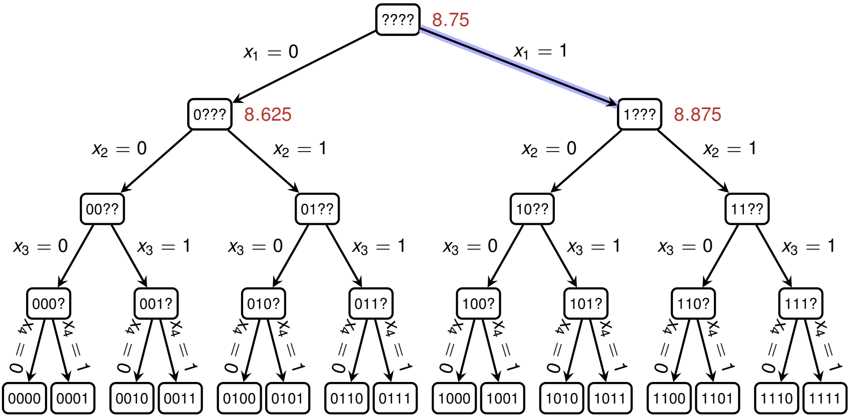
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor \overline{x_4}) \land (x_1 \lor x_2 \lor \overline{x_4}) \land (\overline{x_1} \lor \overline{x_3} \lor x_4) \land (x_1 \lor x_2 \lor \overline{x_4}) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \land (\overline{x_1} \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor x_3) \land (x_1 \lor x_3 \lor x_4) \land (x_2 \lor \overline{x_3} \lor \overline{x_4})$



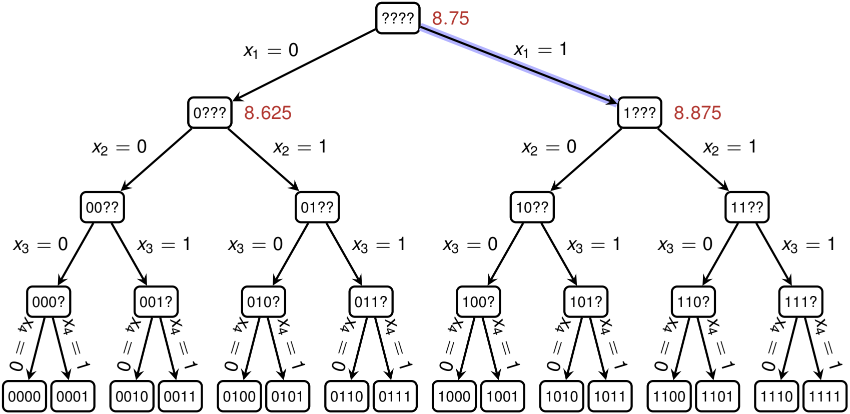
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3) \wedge 1 \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



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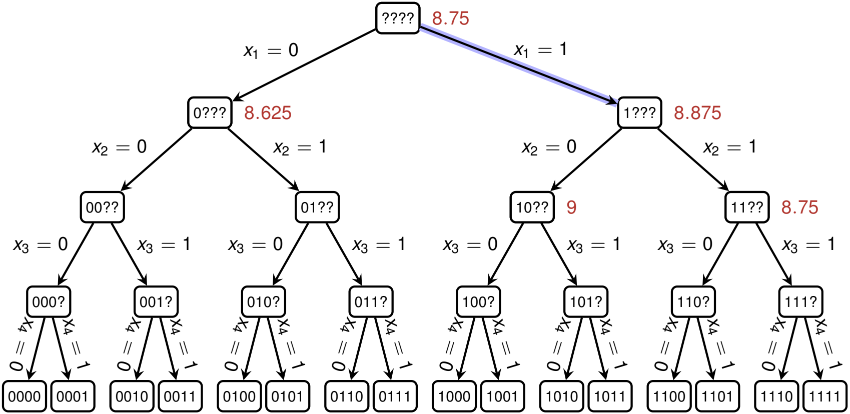

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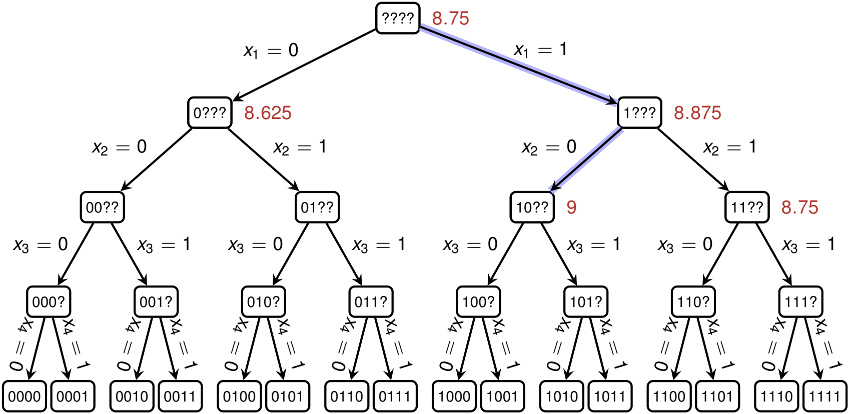

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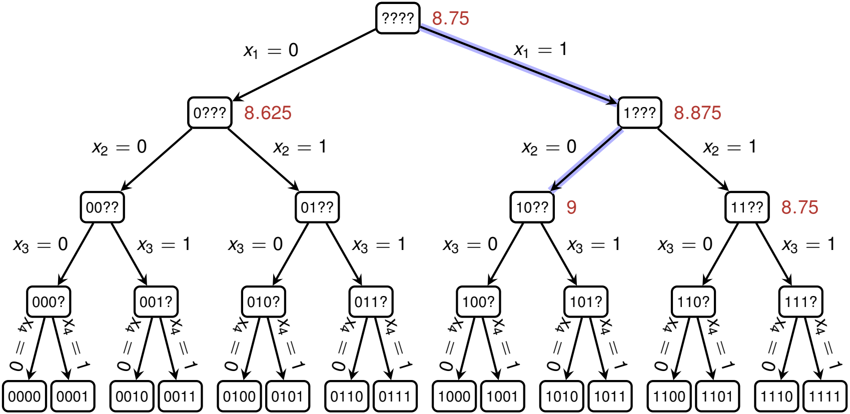
$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge \cancel{(\overline{x_2} \vee \overline{x_3})} \wedge (\cancel{x_2} \vee x_3) \wedge \cancel{(\overline{x_2} \vee \overline{x_3})} \wedge 1 \wedge (\cancel{x_2} \vee \overline{x_3} \vee \overline{x_4})$$



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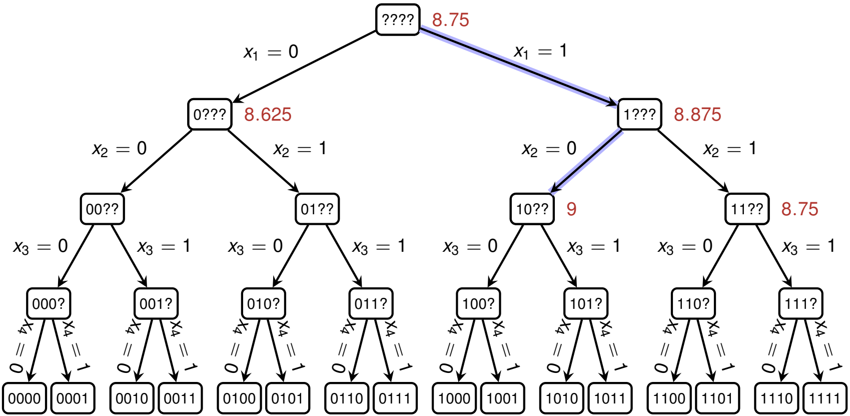

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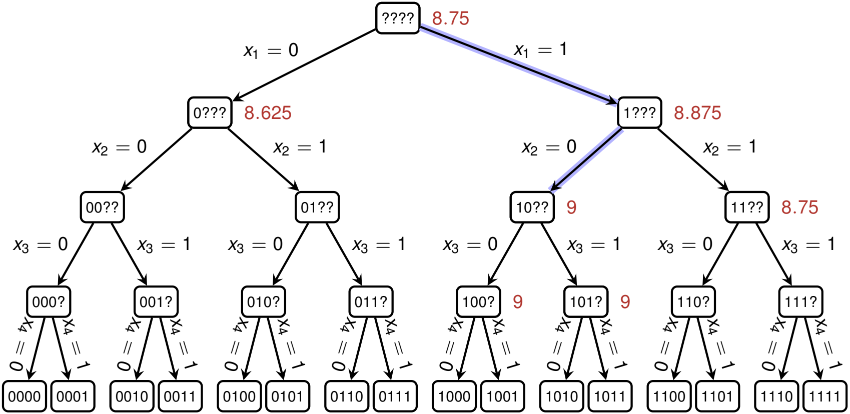

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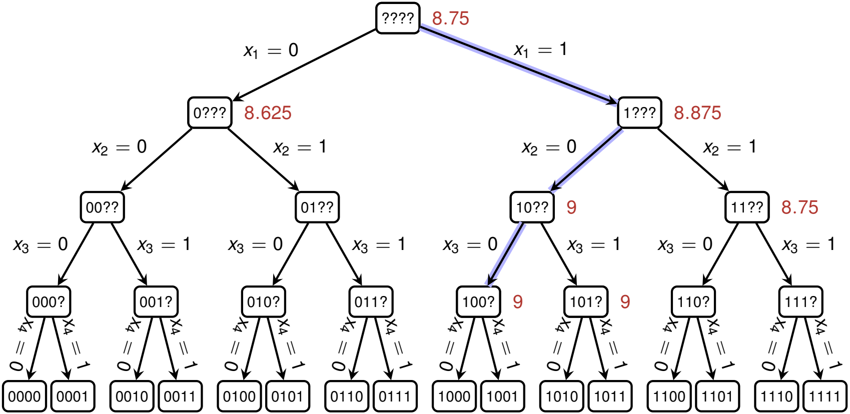
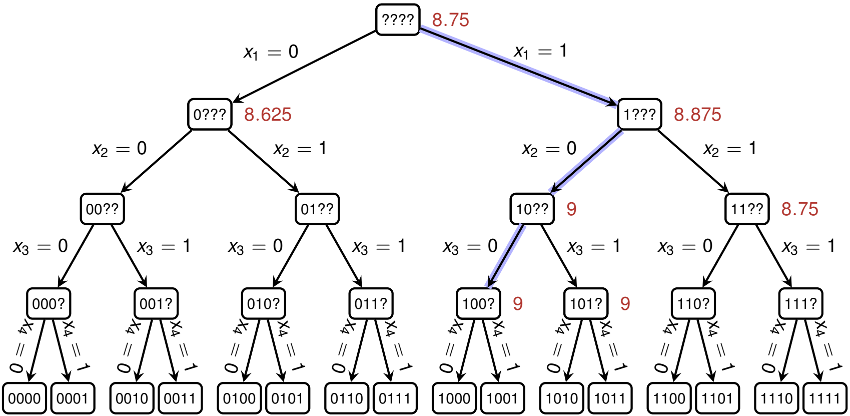


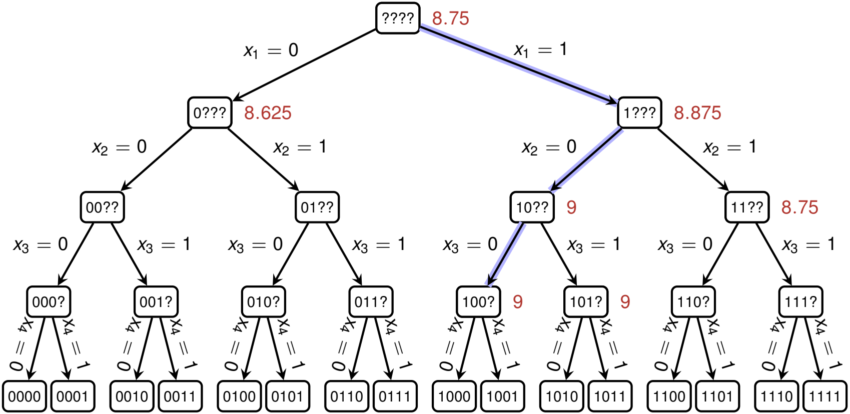
Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee \overline{x_4}) \wedge 1 \wedge 1 \wedge (x_3) \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee \overline{x_4})$



Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



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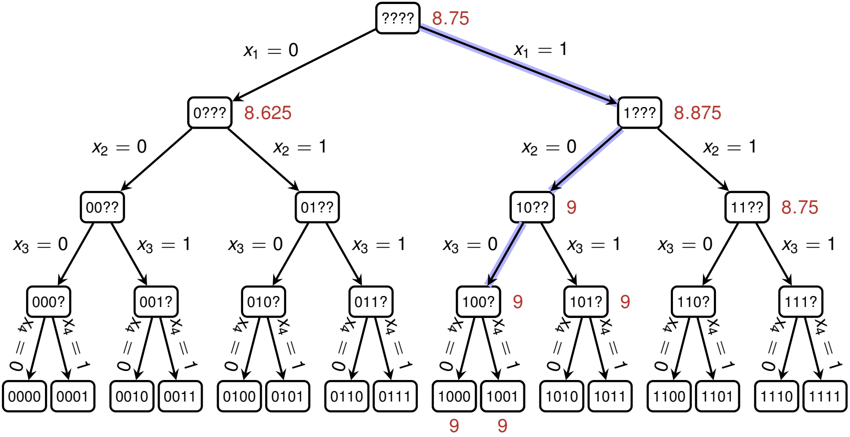

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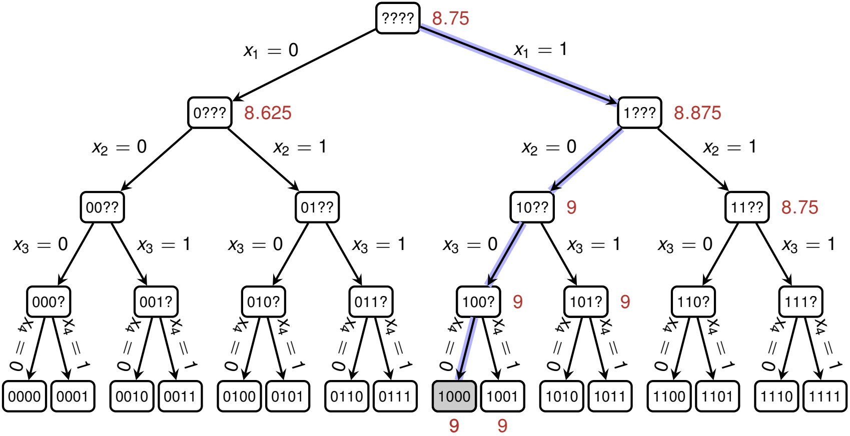

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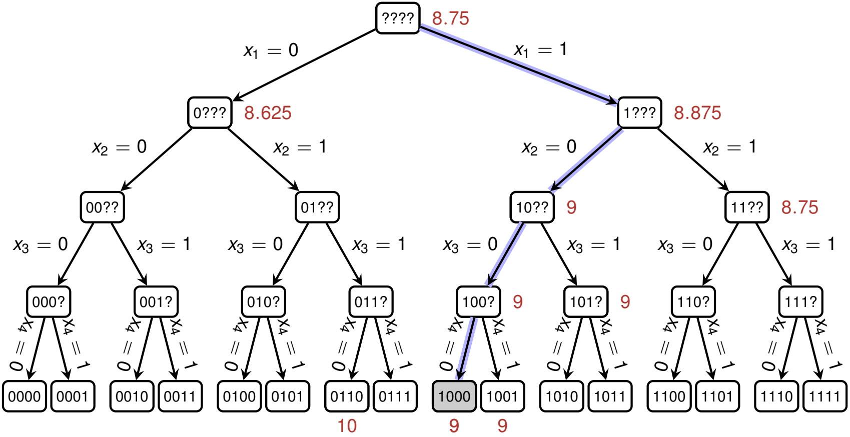

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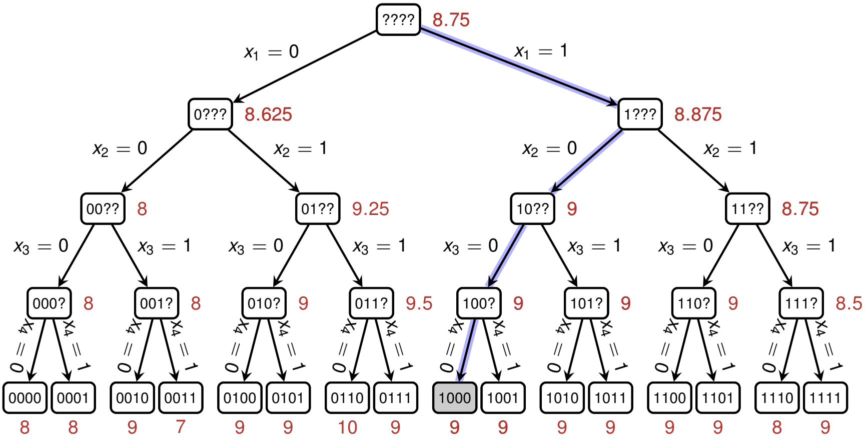

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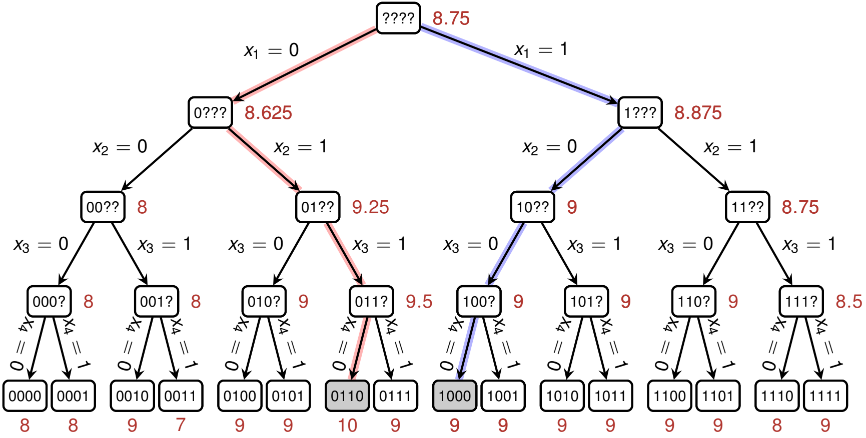

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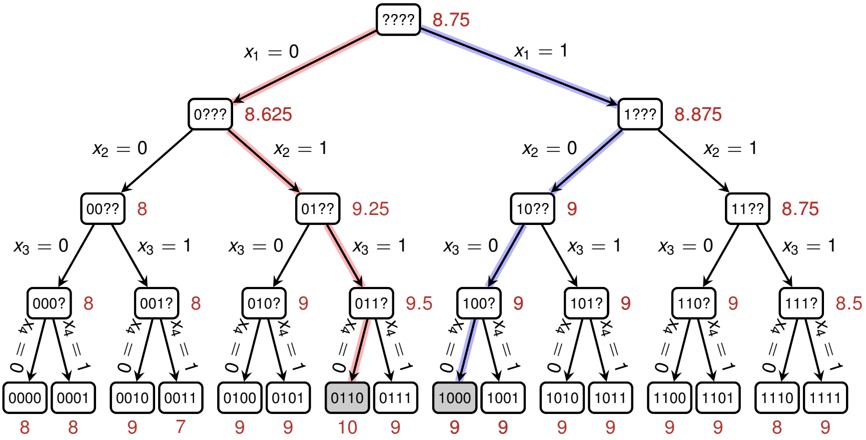

Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised 8/7-approximation algorithm.

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Theorem (Hastad'97)

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-SAT unless P=NP.

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Essentially there is nothing smarter than just guessing!

Outline

Randomised Approximation

MAX-3-CNF

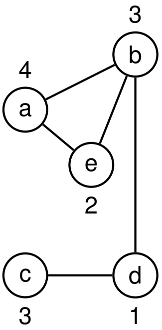
Weighted Vertex Cover

Weighted Set Cover

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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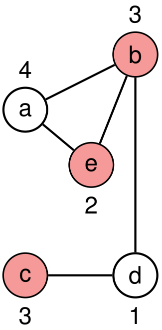

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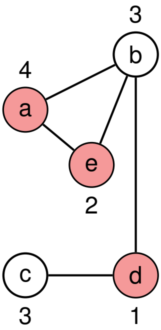

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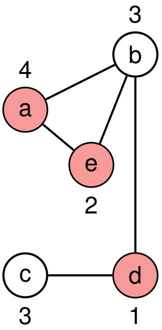

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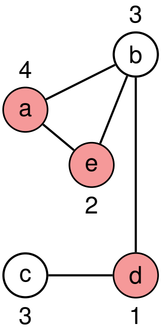

Applications:

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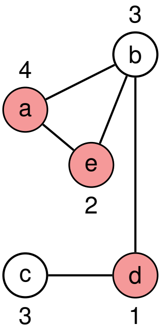

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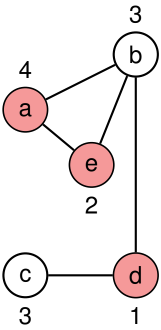

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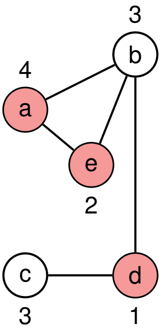

Applications:

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- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

The Greedy Approach from (Unweighted) Vertex Cover

```
APPROX-VERTEX-COVER(G)
C = ∅
E′ = G.E
while E′ ≠ ∅
    let (u, v) be an arbitrary edge of E′
    C = C ∪ {u, v}
    remove from E′ every edge incident on either u or v
return C
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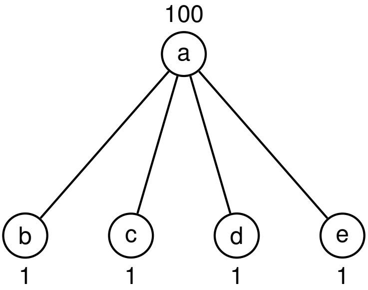
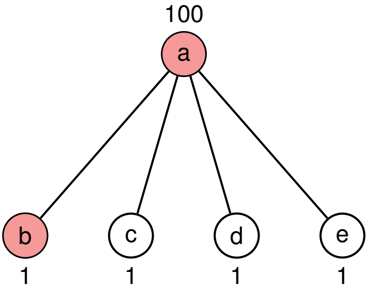


The Greedy Approach from (Unweighted) Vertex Cover

APPROX-VERTEX-COVER(G)

$C = \emptyset$
$E' = G.E$
while $E' \neq \emptyset$
    let (u, v) be an arbitrary edge of E'
    $C = C \cup \{u, v\}$
    remove from E' every edge incident on either u or v
return C



Computed solution has weight 101

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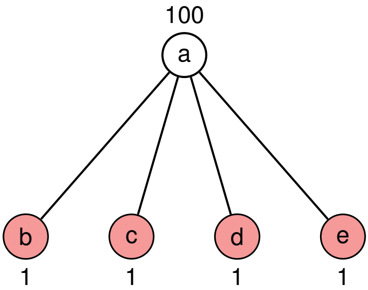

Optimal solution has weight 4

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

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0-1 Integer Program

$$\begin{array}{llcll} \text{minimize} & \displaystyle\sum_{v \in V} w(v)x(v) & & & \\ \text{subject to} & x(u) + x(v) & \geq & 1 & \text{for each } (u, v) \in E \\ & x(v) & \in & \{0, 1\} & \text{for each } v \in V \end{array}$$

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optimum is a lower bound on the optimal weight of a minimum weight-cover.

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Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

The Algorithm

```
APPROX-MIN-WEIGHT-VC(G, w)
C = ∅
compute x̄, an optimal solution to the linear program
for each v ∈ V
    if x̄(v) ≥ 1/2
        C = C ∪ {v}
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APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

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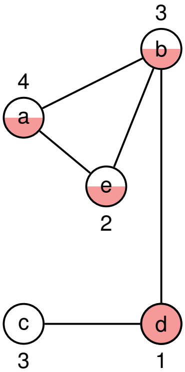
Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

is polynomial-time because we can solve the linear program in polynomial time

Example of APPROX-MIN-WEIGHT-VC

$\overline{x}(a) = \overline{x}(b) = \overline{x}(e) = \frac{1}{2}$, $\overline{x}(d) = 1$, $\overline{x}(c) = 0$



fractional solution of LP
with weight $= 5.5$

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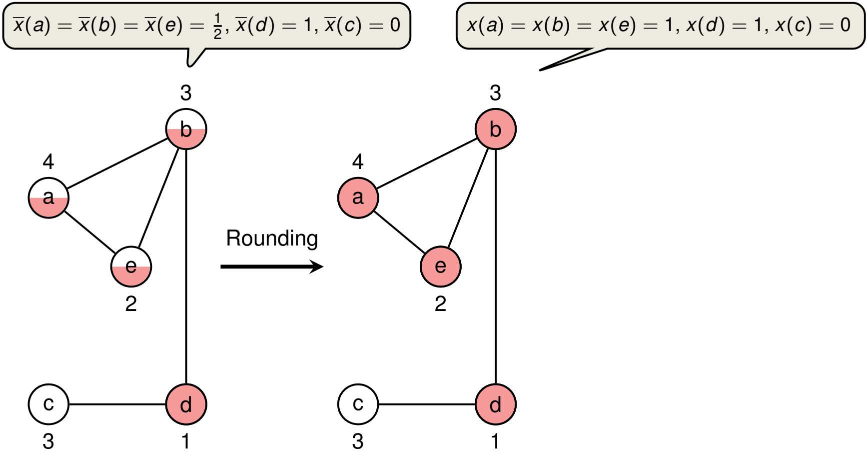

fractional solution of LP
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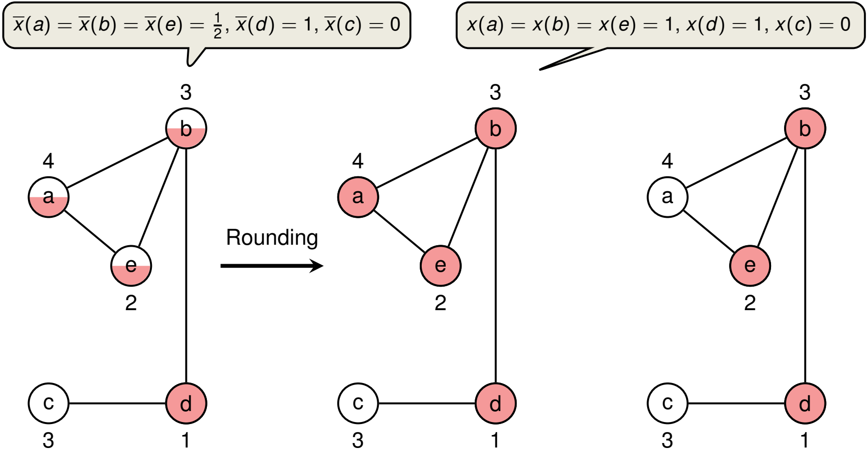

fractional solution of LP with weight $= 5.5$

rounded solution of LP with weight $= 10$

optimal solution with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2):

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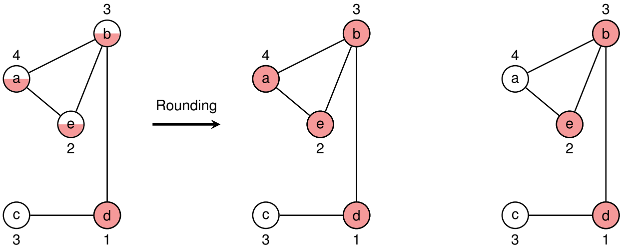

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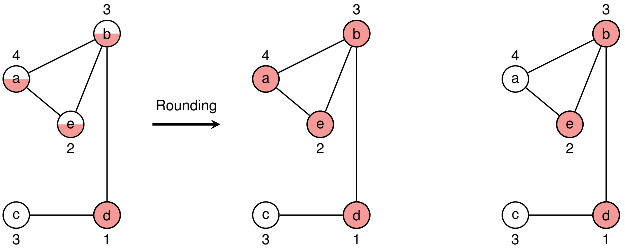

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Proof (Approximation Ratio is 2):

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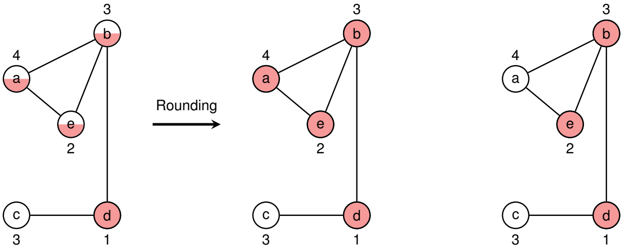

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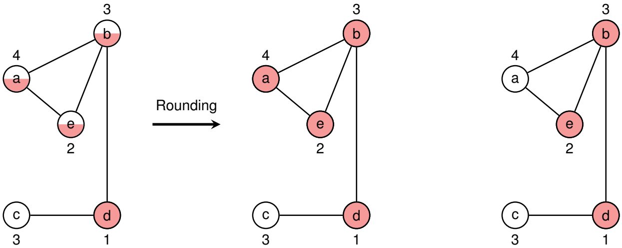

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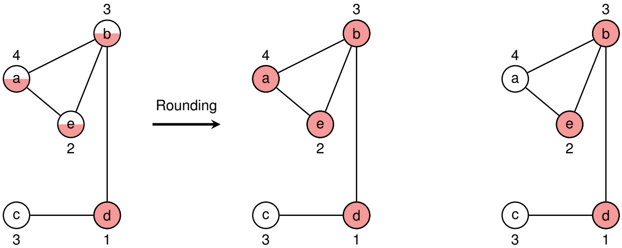

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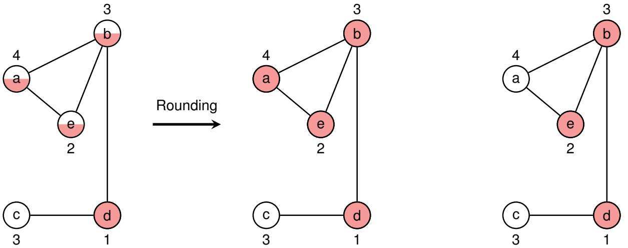

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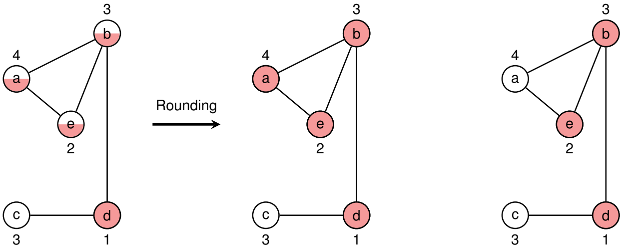

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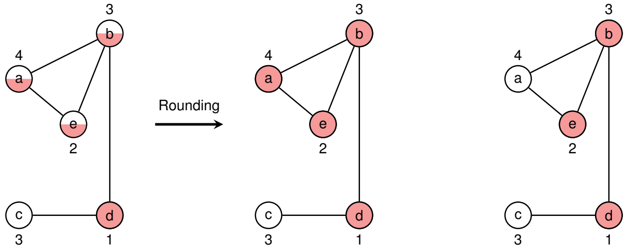

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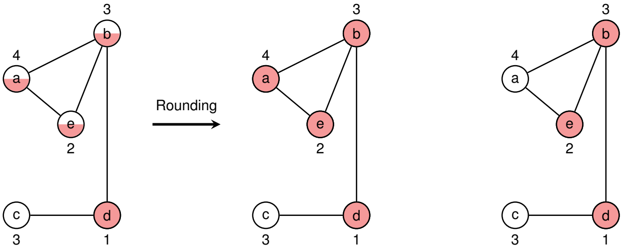

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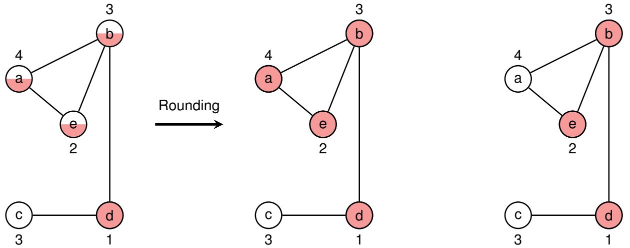

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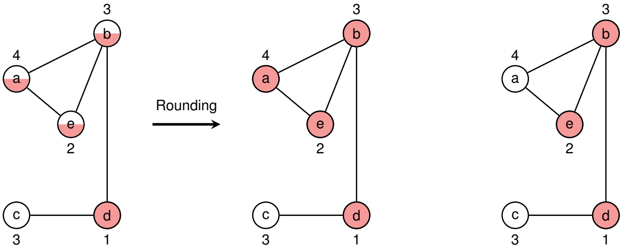

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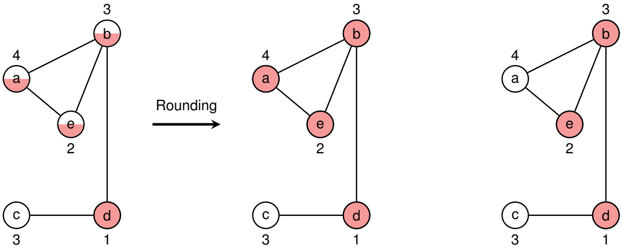

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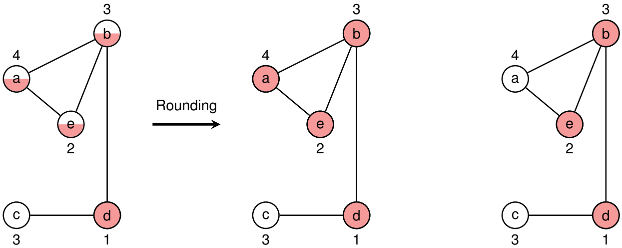

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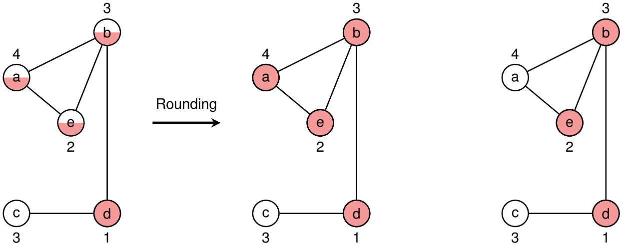

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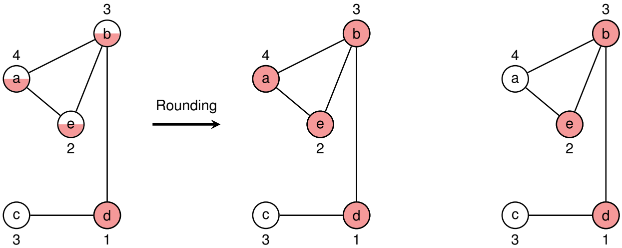

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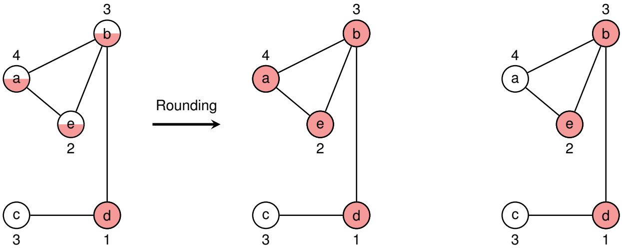

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

The Weighted Set-Covering Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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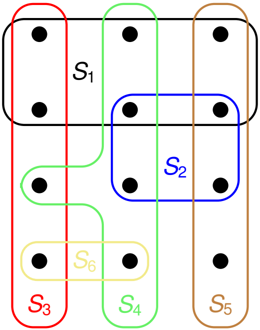

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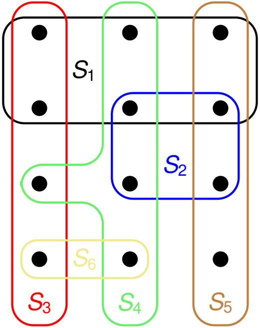

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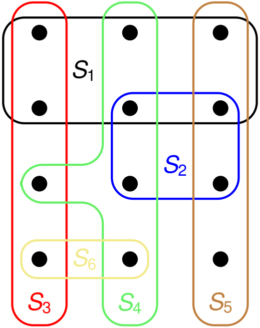

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Remarks:

- generalisation of the weighted vertex-cover problem
- models resource allocation problems

Setting up an Integer Program

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0-1 Integer Program

$$\begin{array}{llll} \text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & \\ \text{subject to} & \displaystyle\sum_{S \in \mathcal{F}\colon x \in S} y(S) \geq 1 & & \text{for each } x \in X \\ & y(S) \in \{0,1\} & & \text{for each } S \in \mathcal{F} \end{array}$$

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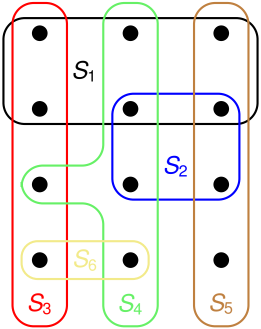

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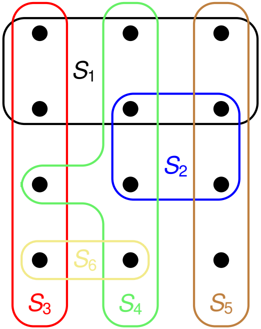

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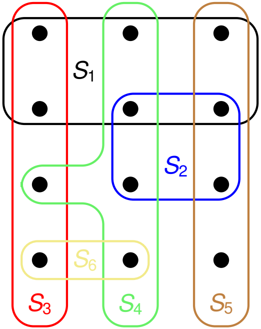

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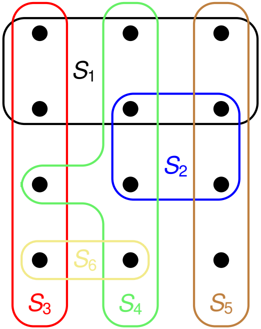

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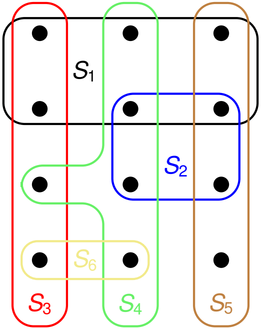

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Even worse: If all y's were below $1/2$, we would not even return a valid cover!

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- Therefore, $\mathbf{E}[\,\bar{y}(S)\,] = y(S)$.

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$$\mathbf{Pr}[\,x \notin \cup_{S \in \mathcal{C}} S\,]$$

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Weighted Set Cover-LP$(X, \mathcal{F}, c)$

compute y, an optimal solution to the linear program
$\mathcal{C} = \emptyset$
repeat $2 \ln n$ times
    for each $S \in \mathcal{F}$
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clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
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Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
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- **Step 1**: The probability that $\mathcal{C}$ is a cover ✓
  - By previous Lemma, an element $x \in X$ is covered in one of the $2\ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that

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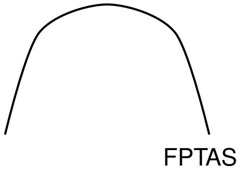
Typical Approach for Designing Approximation Algorithms based on LPs

Spectrum of Approximations



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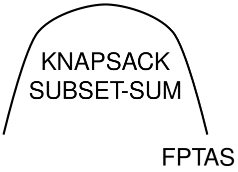

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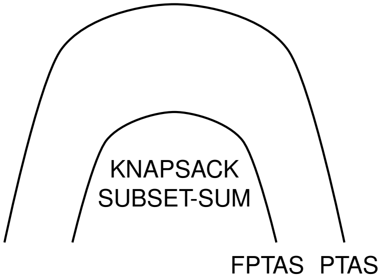

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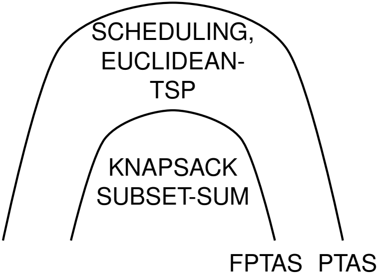

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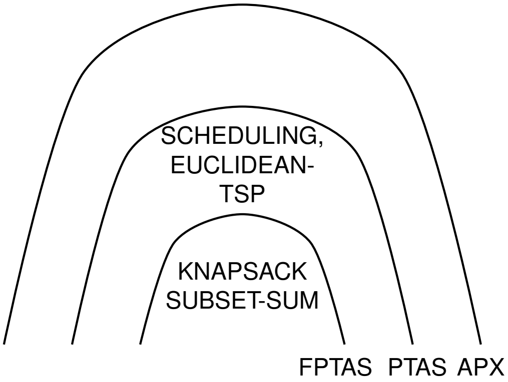

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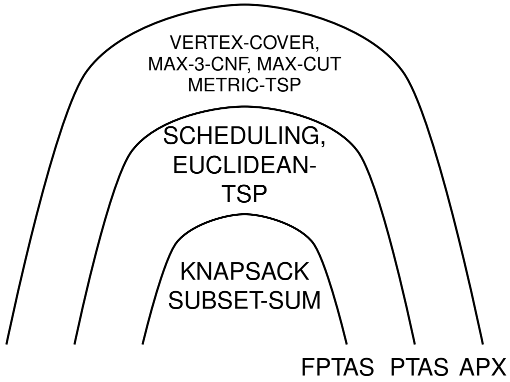

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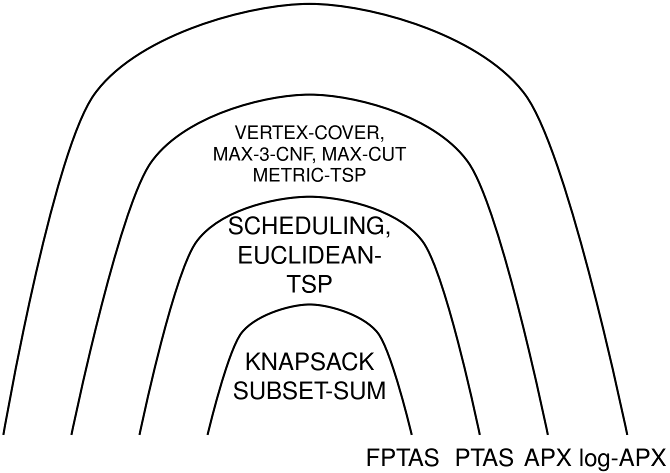

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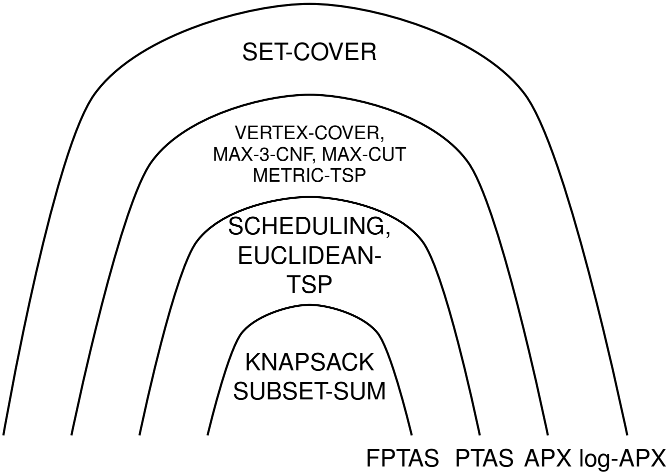

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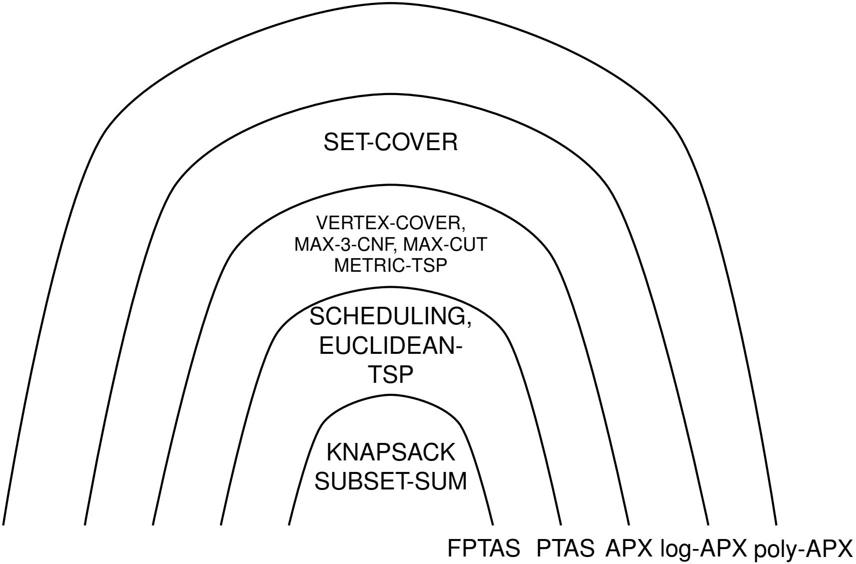

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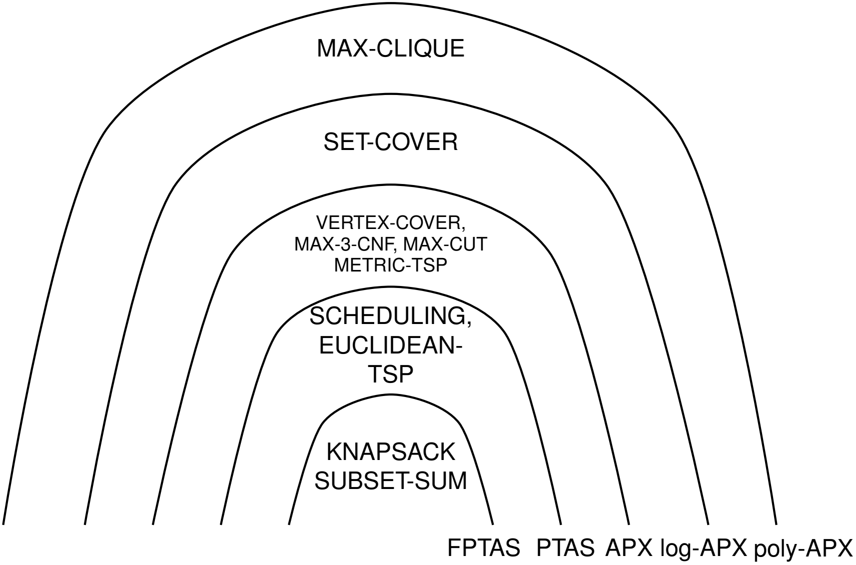

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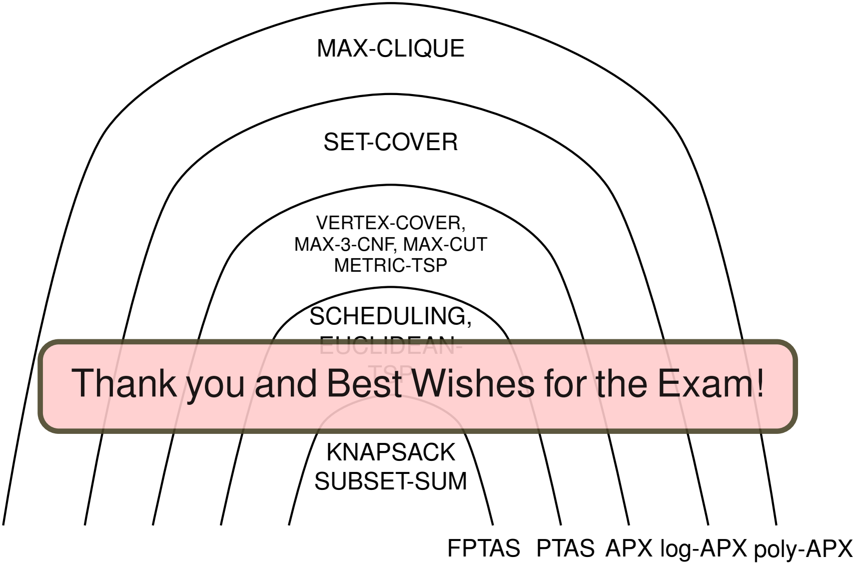

Thank you and Best Wishes for the Exam!