VIII. Approximation Algorithms: MAX-CUT Problem (Outlook)

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UNIVERSITY OF CAMBRIDGE

Outline

Simple Algorithms for MAX-CUT

A Solution based on Semidefinite Programming

Summary

Max-Cut

MAX-CUT Problem

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- Given: Undirected graph $G = (V, E)$

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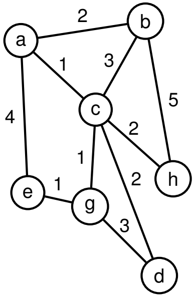

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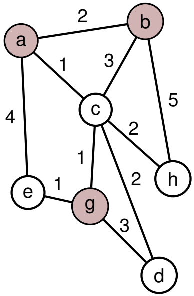

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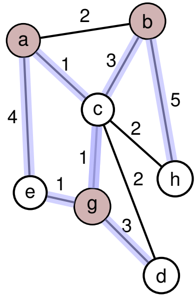

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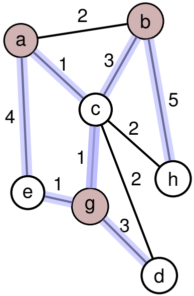

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Applications:

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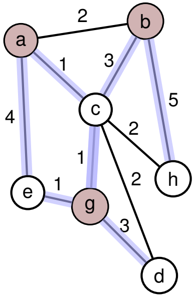

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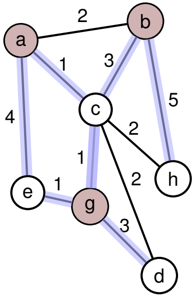

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Ex 35.4-3

Suppose that for each vertex v, we randomly and independently place v in S with probability $1/2$ and in $V \setminus S$ with probability $1/2$. Then this algorithm is a randomized 2-approximation algorithm.

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Suppose that for each vertex v, we randomly and independently place v in S with probability $1/2$ and in $V \setminus S$ with probability $1/2$. Then this algorithm is a randomized 2-approximation algorithm.

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We could employ the same derandomisation used for MAX-3-CNF.

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Local Search

Local Search: Switch side of a vertex if it increases the cut.

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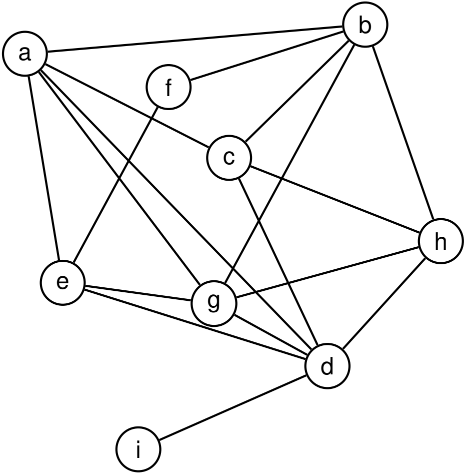
LOCAL SEARCH(G, w)

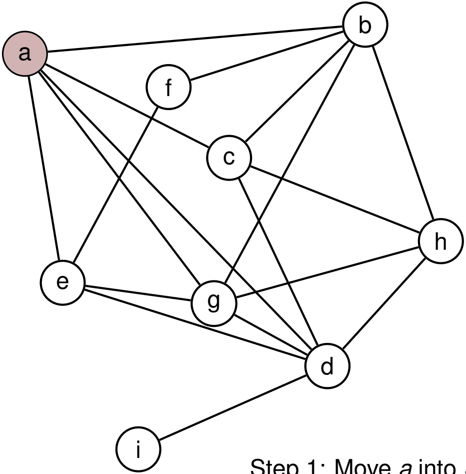
Let S be an arbitrary subset of V
do
    $flag = 0$
    if $\exists u \in S$ with $w(S \setminus \{u\}, (V \setminus S) \cup \{u\}) \geq w(S, V \setminus S)$ then
        $S = S \setminus \{u\}$
        $flag = 1$
    end if
    if $\exists u \in V \setminus S$ with $w(S \cup \{u\}, (V \setminus S) \setminus \{u\}) \geq w(S, V \setminus S)$ then
        $S = S \cup \{u\}$
        $flag = 1$
    end if
while $flag = 1$
return S

Illustration of Local Search



Cut $= 0$

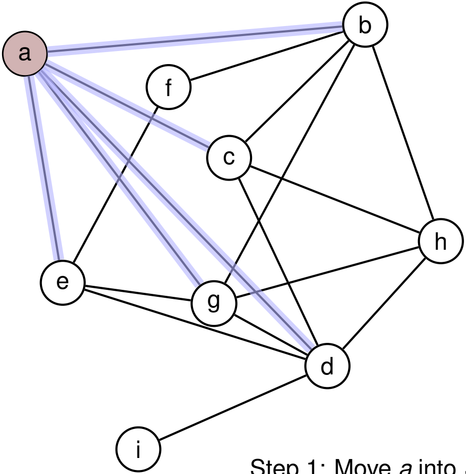
Illustration of Local Search



Step 1: Move a into S

$\text{Cut} = 0$

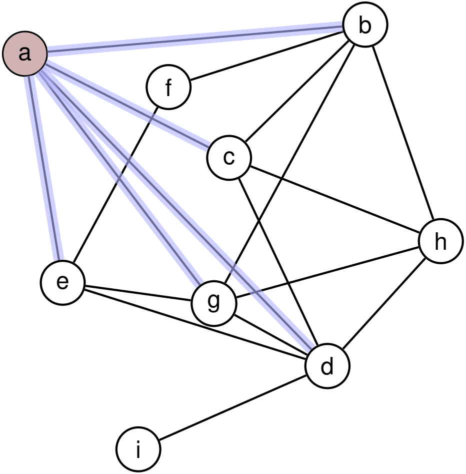
Illustration of Local Search



Step 1: Move *a* into *S*

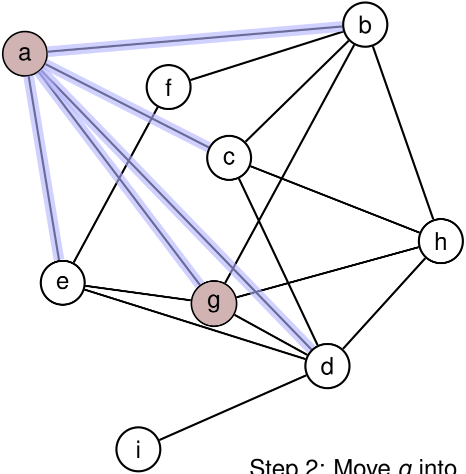
$\text{Cut} = \cancel{0}$

Illustration of Local Search



Cut $= 5$

Illustration of Local Search



Step 2: Move g into S

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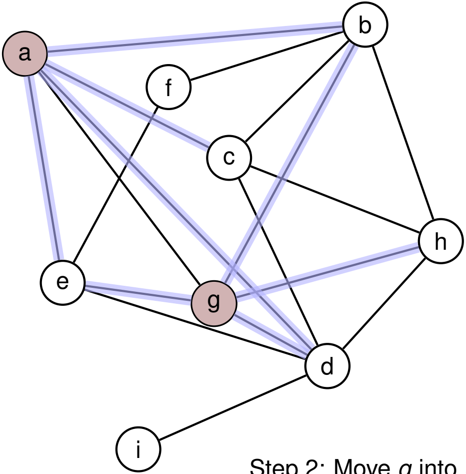
Illustration of Local Search



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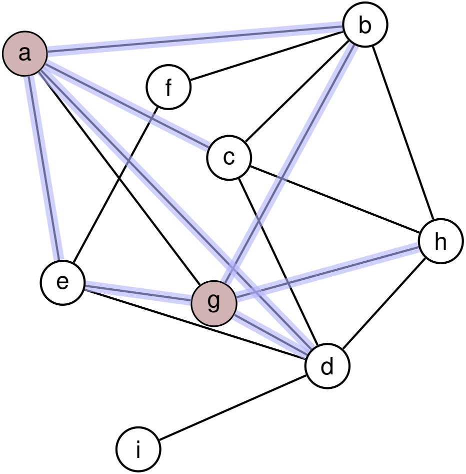
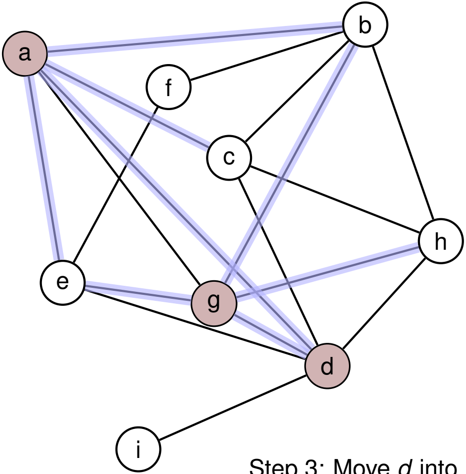


Cut = 8

Illustration of Local Search



Step 3: Move d into S

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Illustration of Local Search

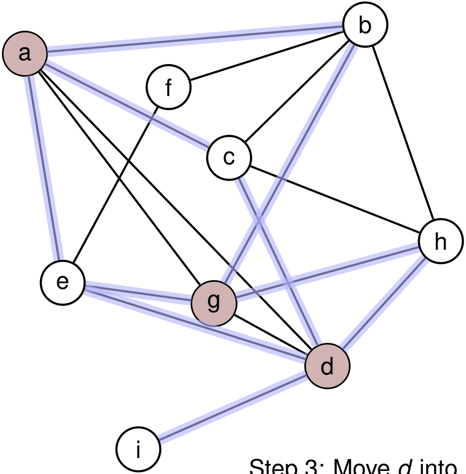

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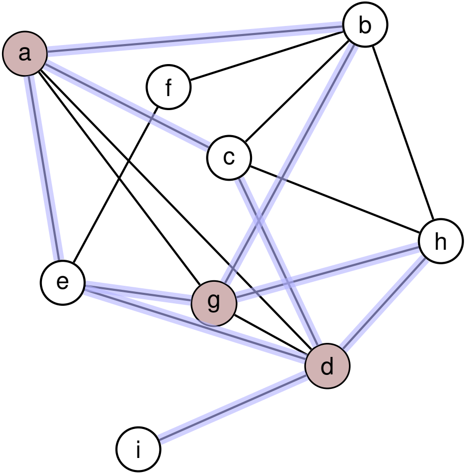
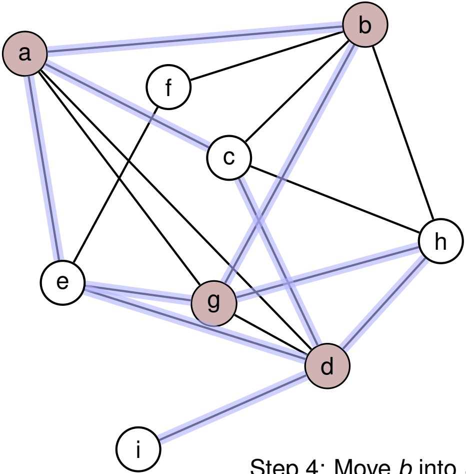
Illustration of Local Search



Cut $= 10$

Illustration of Local Search



Step 4: Move *b* into *S*

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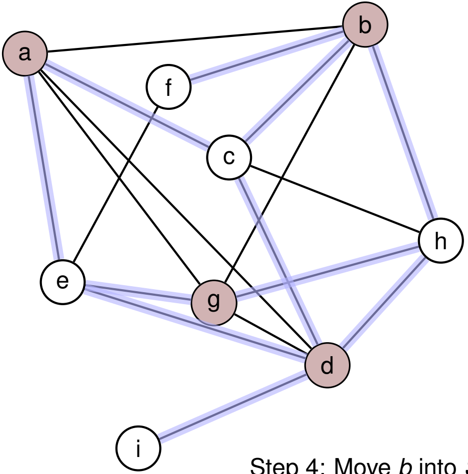
Illustration of Local Search



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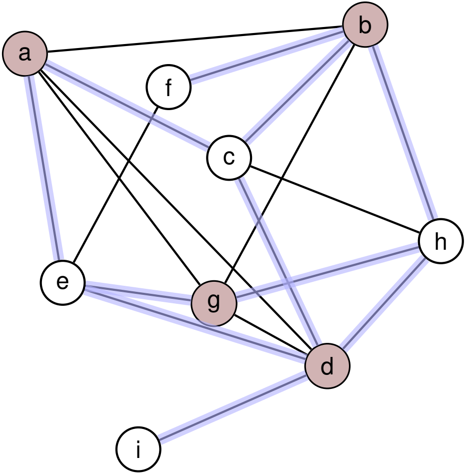
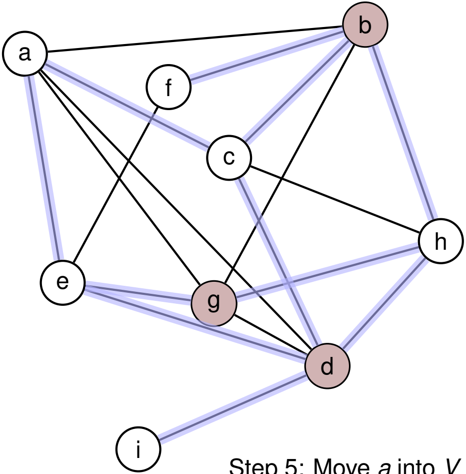


Cut $= 11$

Illustration of Local Search



Step 5: Move a into $V \setminus S$

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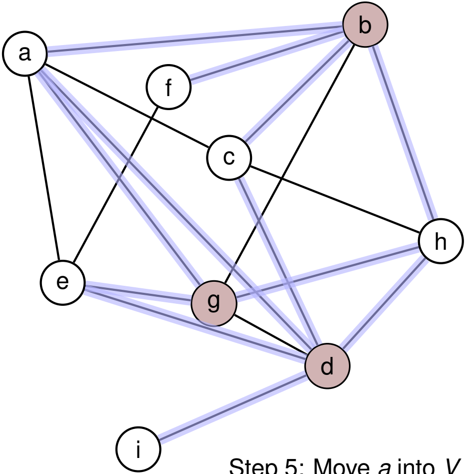

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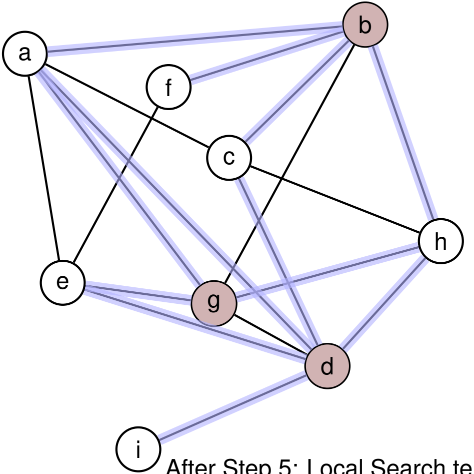

After Step 5: Local Search terminates

$\text{Cut} = 12$

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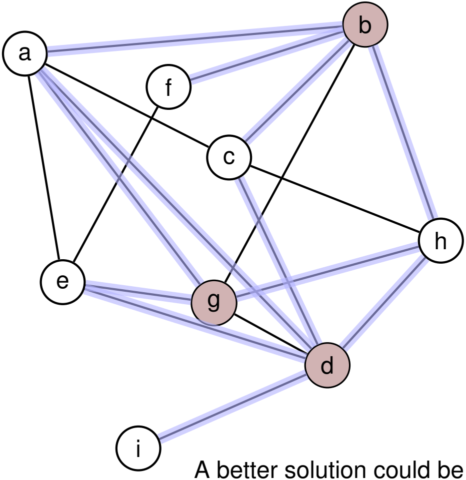

A better solution could be found:

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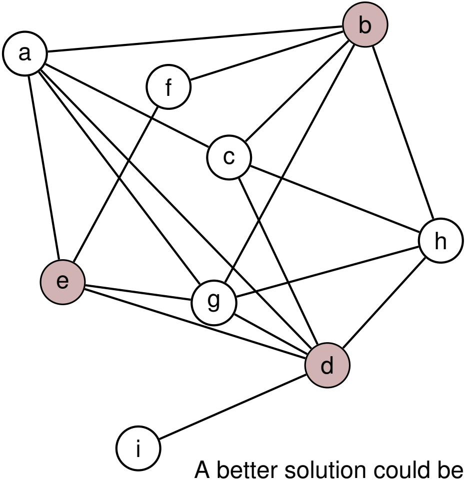

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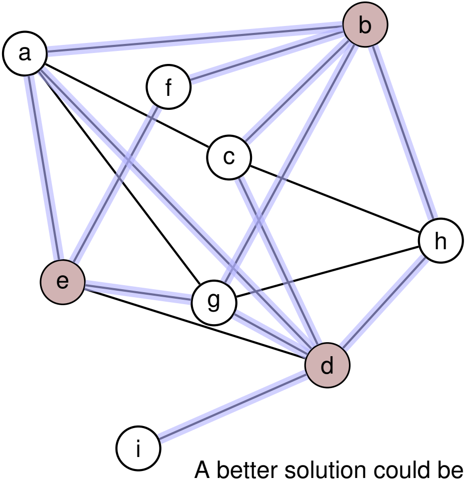

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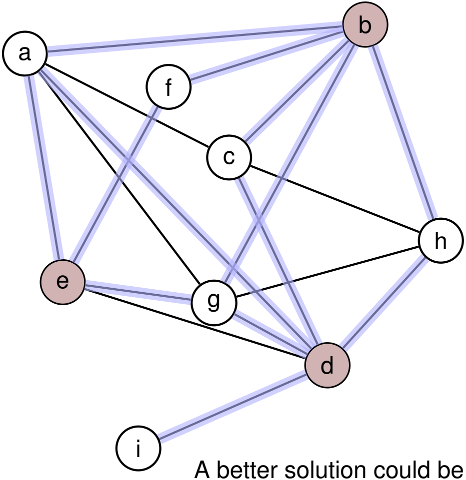

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Analysis of Local Search (1/2)

Theorem

The cut returned by LOCAL-SEARCH satisfies $W \geq (1/2)W^*$.

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The cut returned by LOCAL-SEARCH satisfies $W \geq (1/2)W^*$.

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Every edge appears on one of the two sides.

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- **Unweighted Graphs:** Cut increases by at least one in each iteration $\Rightarrow$ at most n^2 iterations
- **Weighted Graphs:** could take exponential time in n (not obvious...)

Outline

Simple Algorithms for MAX-CUT

A Solution based on Semidefinite Programming

Summary

Max-Cut Problem

High-Level-Approach:

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$$\begin{array}{lll} \text{maximize} & \displaystyle\frac{1}{2}\sum_{(i,j)\in E} w_{i,j}\cdot(1-y_iy_j) & \\ \text{subject to} & y_i\in\{-1,+1\}, & i=1,\ldots,n. \end{array}$$

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Label vertices by $1, 2, \ldots, n$ and express weight function etc. as a $n \times n$-matrix.

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$$S = \{i \in V : y_i = +1\},$$
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Any solution of the original program can be recovered by setting $v_i = (y_i, 0, 0, \ldots, 0)$!

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Reformulating the Quadratic Program as a Semidefinite Program

Vector Programming Relaxation

$$\begin{array}{lll} \text{maximize} & \displaystyle \frac{1}{2} \sum_{(i,j) \in E} w_{i,j} \cdot (1 - v_i v_j) & \\ \text{subject to} & v_i \cdot v_i = 1 & i = 1, \ldots, n. \\ & v_i \in \mathbb{R}^n & \end{array}$$

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Solve this (which can be done in polynomial time), and recover V using Cholesky Decomposition.

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1. Pick a random vector $r=(r_1,r_2,\ldots,r_n)$ by drawing each component from $\mathcal{N}(0,1)$

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The probability that two vectors $v_i, v_j\in\mathbb{R}^n$ are separated by the (random) hyperplane given by r equals $\frac{\arccos(v_i\cdot v_j)}{\pi}$.

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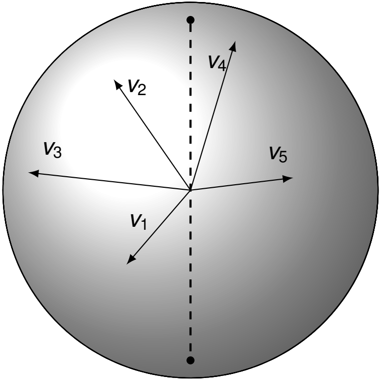
Follows by projecting on the plane given by v_i and v_j.

Illustration of the Hyperplane

N



S

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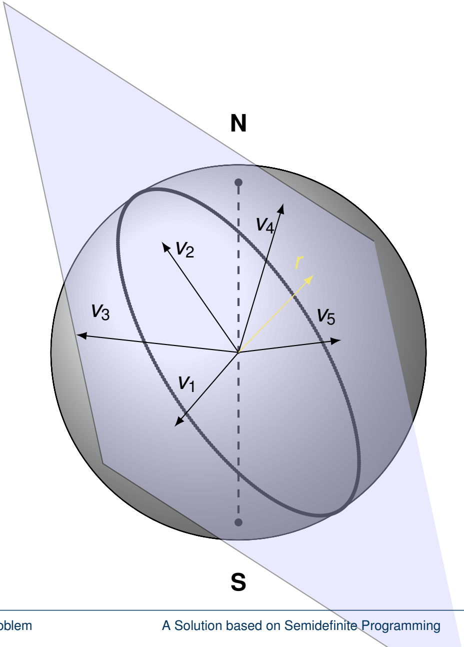

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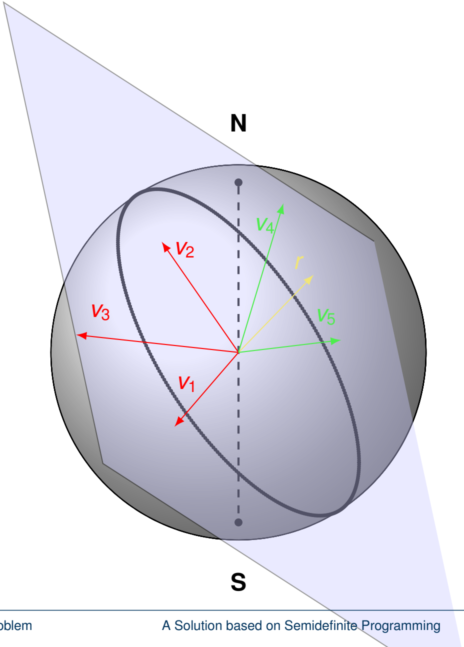

A second (technical) Lemma

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For any $x \in [-1, 1]$,

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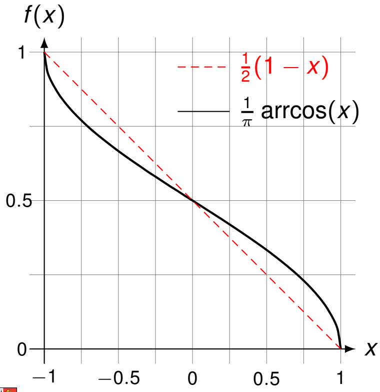

A second (technical) Lemma

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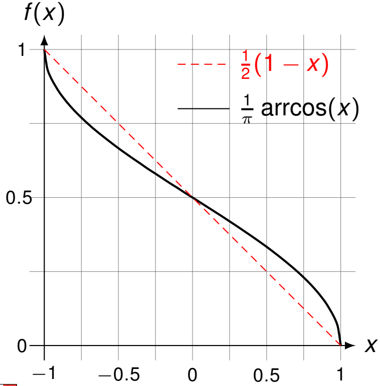

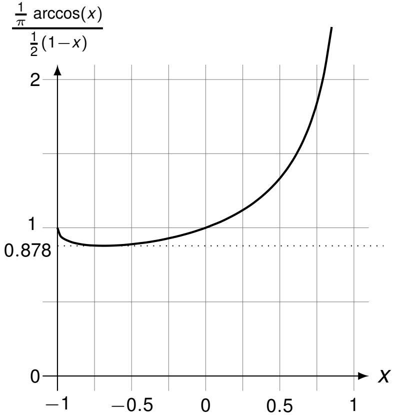

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Theorem (Khot, Kindler, Mossel, O'Donnell'04)

Assuming the so-called Unique Games Conjecture holds, unless P=NP there is no ρ-approximation algorithm for MAX-CUT with

$$\rho \leq \max_{-1 \leq x \leq 1} \frac{\frac{1}{2}(1-x)}{\frac{1}{\pi}\arccos(x)} \leq 1.139$$

Other Approximation Algorithms for MAX-CUT

Theorem (Mathieu, Schudy'08)

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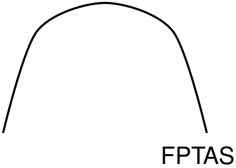
Outline

Simple Algorithms for MAX-CUT

A Solution based on Semidefinite Programming

Summary

Spectrum of Approximations



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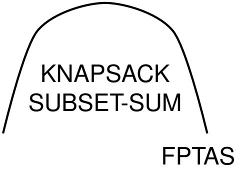

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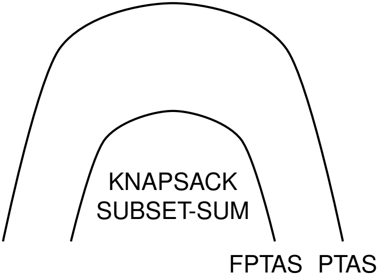

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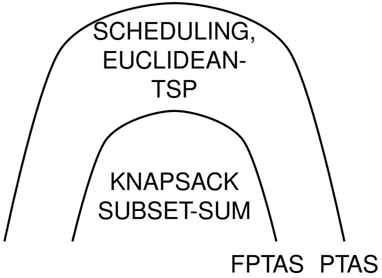

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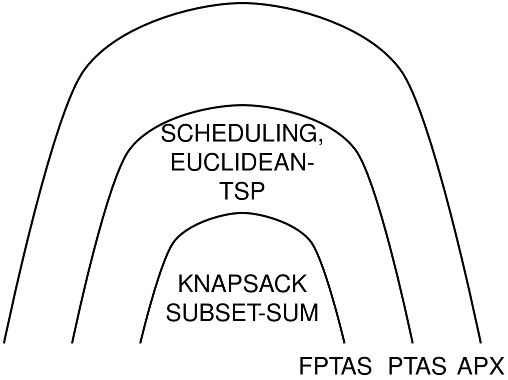

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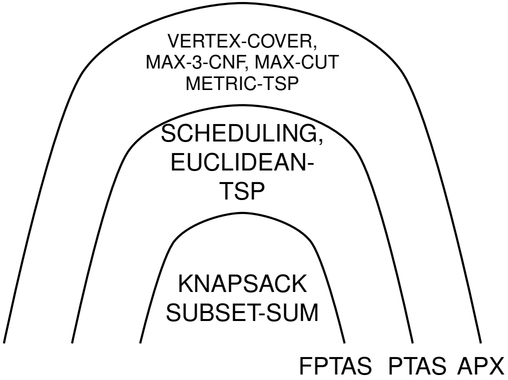

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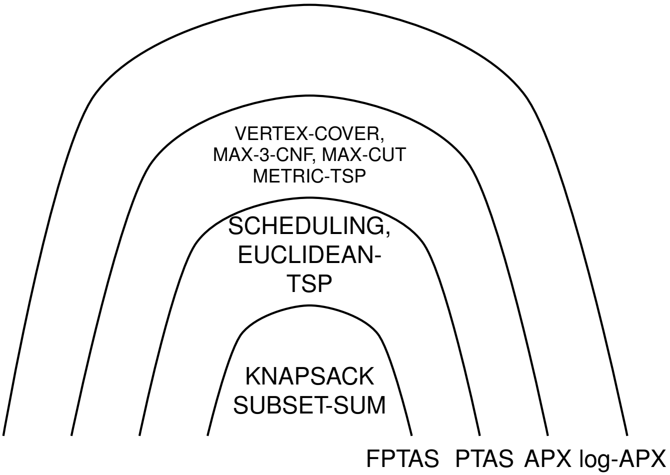

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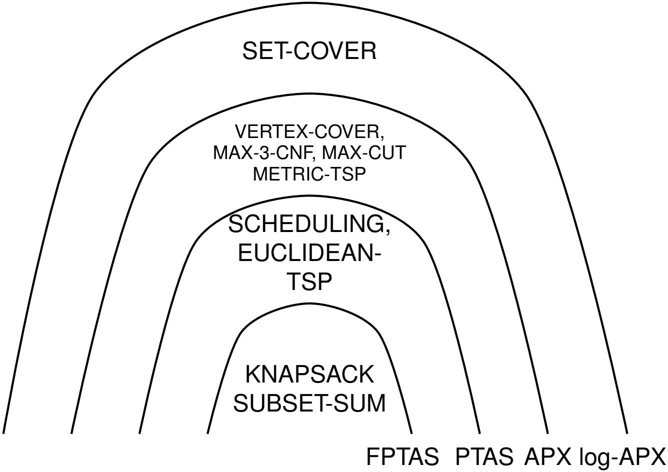

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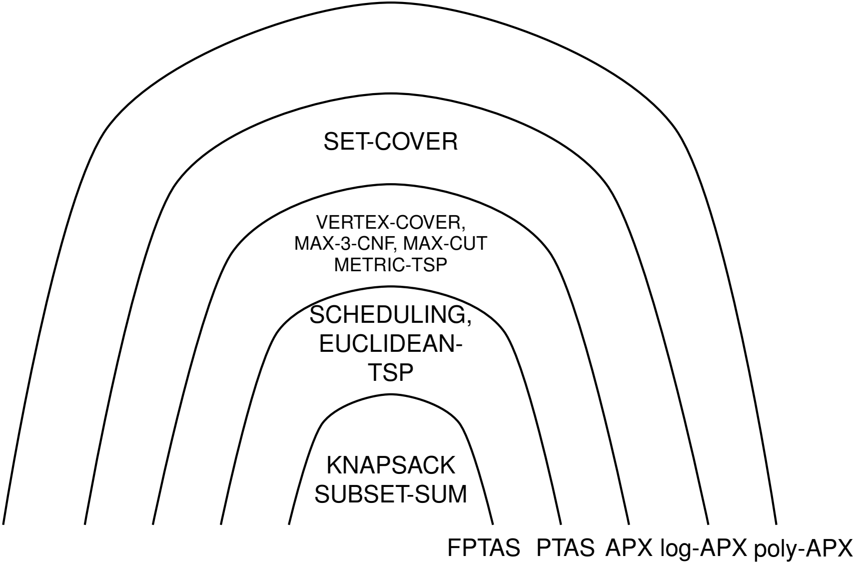

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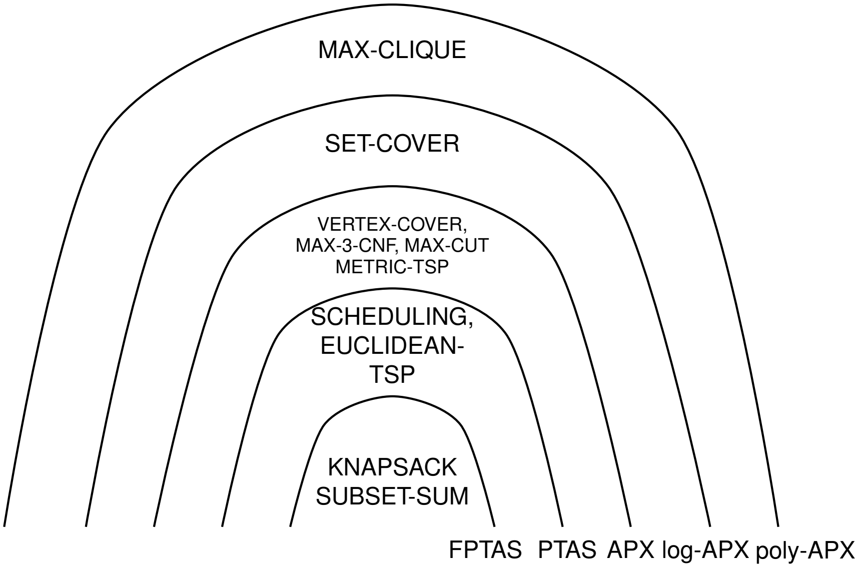

The End

Thank you very much and
Best Wishes for the Exam!