II. Matrix Multiplication

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Outline

Matrix Multiplication

Remember: If $A = (a_{ij})$ and $B = (b_{ij})$ are square $n \times n$ matrices, then the matrix product $C = A \cdot B$ is defined by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

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SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
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SQUARE-MATRIX-MULTIPLY(A, B) takes time $\Theta(n^3)$.

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$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \forall i, j = 1, 2, \ldots, n.$$

This definition suggests that $n^2 \cdot n = n^3$ arithmetic operations are necessary.

SQUARE-MATRIX-MULTIPLY(A, B)

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Outline

Introduction

Serial Matrix Multiplication

Digression: Multithreading

Multithreaded Matrix Multiplication

Divide & Conquer: First Approach

Assumption: n is always an exact power of 2.

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Partition A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

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Hence the equation $C = A \cdot B$ becomes:

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Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

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This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Divide & Conquer: First Approach

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Partition A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Hence the equation $C = A \cdot B$ becomes:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

This corresponds to the four equations:

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\ C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{aligned}$$

Each equation specifies two multiplications of $n/2 \times n/2$ matrices and the addition of their products.

Divide & Conquer: First Approach (Pseudocode)

$$
\begin{aligned}
C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\
C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\
C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21} \\
C_{11} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22}
\end{aligned}
$$

Divide & Conquer: First Approach (Pseudocode)

```
SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
n = A.rows
let C be a new n × n matrix
if n == 1
    c11 = a11 · b11
else partition A, B, and C as in equations (4.9)
    C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B21)
    C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A12, B22)
    C21 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B11)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B21)
    C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
          + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
return C
```

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
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Divide & Conquer: First Approach (Pseudocode)

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SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)
1   n = A.rows
2   let C be a new n × n matrix
3   if n == 1
4       c11 = a11 · b11
5   else partition A, B, and C as in equations (4.9)
6       C11 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B11)
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7       C12 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A11, B12)
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9       C22 = SQUARE-MATRIX-MULTIPLY-RECURSIVE(A21, B12)
              + SQUARE-MATRIX-MULTIPLY-RECURSIVE(A22, B22)
10  return C
```

Line 5: Handle submatrices implicitly through index calculations instead of creating them.

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$$
$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$$
$$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$$
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Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

$n = A.rows$
let C be a new $n \times n$ matrix
if $n == 1$
  $c_{11} = a_{11} \cdot b_{11}$
else partition A, B, and C as in equations (4.9)
  $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
    $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
  $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
    $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
  $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
    $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
  $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
    $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
return C

Let $T(n)$ be the runtime of this procedure.

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return C

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ & \text{if } n > 1. \end{cases}$$

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8 Multiplications

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return C

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

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n = A.rows
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8 Multiplications

4 Additions and Partitioning

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return C
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Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

8 Multiplications

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Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) =$

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Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n})$

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Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

No improvement over the naive algorithm!

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

```
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Let $T(n)$ be the runtime of this procedure. Then:

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Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

Divide & Conquer: First Approach (Pseudocode)

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

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  $C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11})
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  $C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12})
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  $C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11})
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  $C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12})
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return C

Let $T(n)$ be the runtime of this procedure. Then:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ 8 \cdot T(n/2) + \Theta(n^2) & \text{if } n > 1. \end{cases}$$

Solution: $T(n) = \Theta(8^{\log_2 n}) = \Theta(n^3)$

Goal: Reduce the number of multiplications

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Divide & Conquer: Second Approach

Idea: Make the recursion tree less bushy by performing only **7** recursive multiplications of $n/2 \times n/2$ matrices.

Strassen's Algorithm (1969)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices
2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$. Each is $n/2 \times n/2$ and is the sum or difference of two matrices created in the previous step.
3. Recursively compute 7 matrix products $P_1, P_2, \ldots, P_7$, each $n/2 \times n/2$
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Divide & Conquer: Second Approach

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Time for steps 1,2,4: $\Theta(n^2)$, hence $T(n) = 7 \cdot T(n/2) + \Theta(n^2) \Rightarrow T(n) = \Theta(n^{\log 7})$.

Solving the Recursion

$T(n) = 7 \cdot T(n/2) + c \cdot n^2$

Details of Strassen's Algorithm

The 10 Submatrices and 7 Products

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\begin{aligned}
P_1 &= A_{11} \cdot S_1 = A_{11} \cdot (B_{12} - B_{22}) \\
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&- \cancel{A_{11}B_{22}} - \cancel{A_{12}B_{22}} + A_{12}B_{21} + \cancel{A_{12}B_{22}} - \cancel{A_{22}B_{21}} - \cancel{A_{22}B_{22}}
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Current State-of-the-Art

Open Problem: Is there an algorithm with quadratic complexity?

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- $O(n^{2.517})$, Romani (1982)
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- $O(n^{2.3728642})$, V. Williams (2011)
- $O(n^{2.3728639})$, Le Gall (2014)
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Outline

Memory Models

Distributed Memory

- Each processor has its private memory
- Access to memory of another processor via messages

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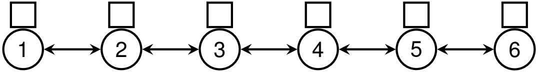

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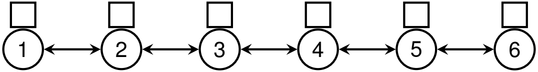

Shared Memory

- Central location of memory
- Each processor has direct access

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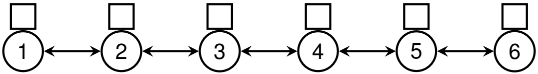

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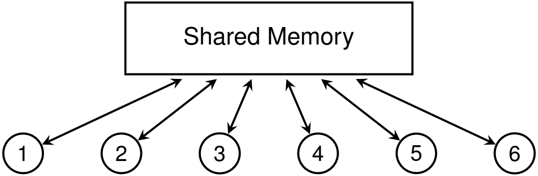

Dynamic Multithreading

- Programming shared-memory parallel computer difficult

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- Use concurrency platform which coordinates all resources

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Scheduling jobs, communication protocols, load balancing etc.

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Functionalities:

Dynamic Multithreading

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Functionalities:

- **`spawn`**

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  - (optional) prefix to a procedure call statement
  - procedure is executed in a separate thread
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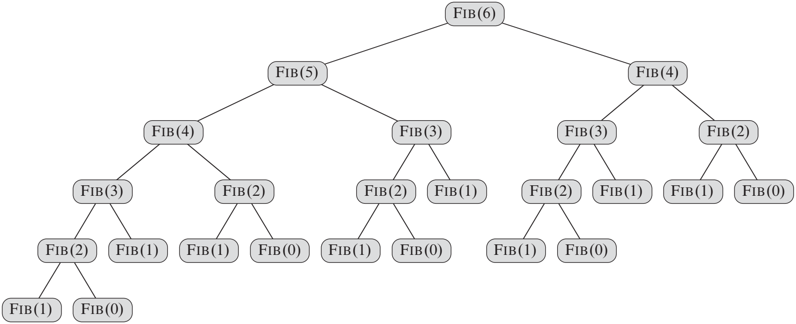
  - each iteration is called in its own thread

Only logical parallelism, but not actual!
Need a scheduler to map threads to processors.

Computing Fibonacci Numbers Recursively (Fig. 27.1)

```
FIB(n)
  if n<=1 return n
  else x=FIB(n-1)
       y=FIB(n-2)
       return x+y
```

Computing Fibonacci Numbers Recursively (Fig. 27.1)

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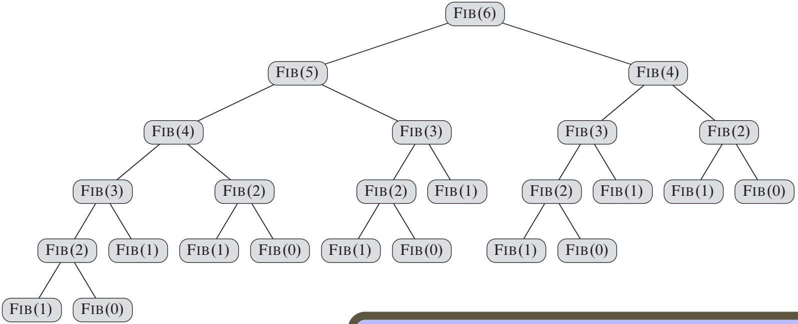

Very inefficient – exponential time!

```
FIB(n)
  if n<=1 return n
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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

```
P-FIB(n)
  if n<=1 return n
  else x=spawn P-FIB(n-1)
       y=P-FIB(n-2)
       sync
       return x+y
```

Computing Fibonacci Numbers in Parallel (Fig. 27.2)

- Without **spawn** and **sync** same pseudocode as before
- **spawn** does not imply parallel execution (depends on scheduler)

```
P-FIB(n)
  if n<=1 return n
  else x=spawn P-FIB(n-1)
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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

Computation Dag $G = (V, E)$

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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

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- V set of threads (instructions/strands without parallel control)

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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

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Computing Fibonacci Numbers in Parallel (Fig. 27.2)

Computation Dag $G = (V, E)$

- V set of threads (instructions/strands without parallel control)
- E set of dependencies

```
0: P-FIB(n)
1:   if n<=1 return n
2:   else x=spawn P-FIB(n-1)
3:        y=P-FIB(n-2)
4:        sync
5:        return x+y
```

Computing Fibonacci Numbers in Parallel (Fig. 27.2)

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P-FIB(n)
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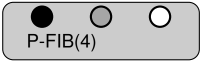


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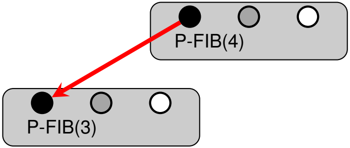
Computing Fibonacci Numbers in Parallel (Fig. 27.2)

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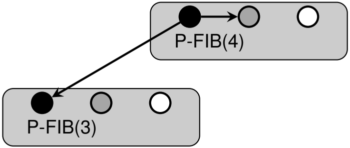


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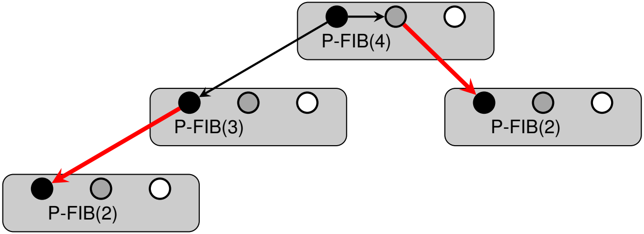


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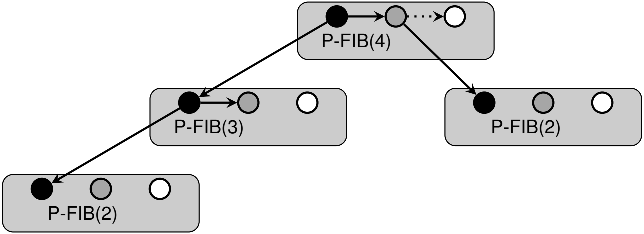


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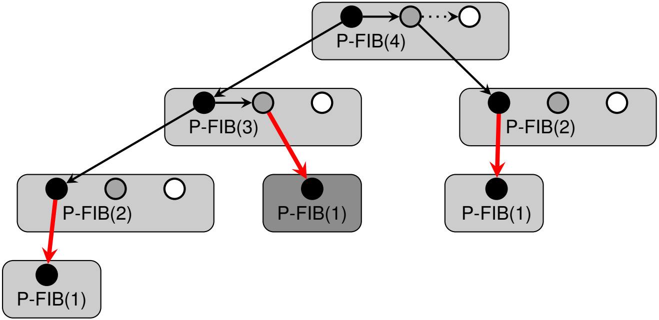


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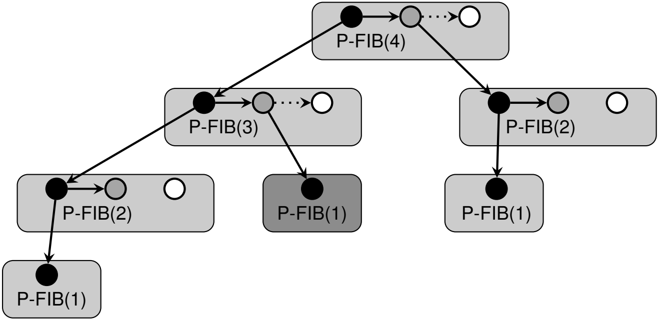


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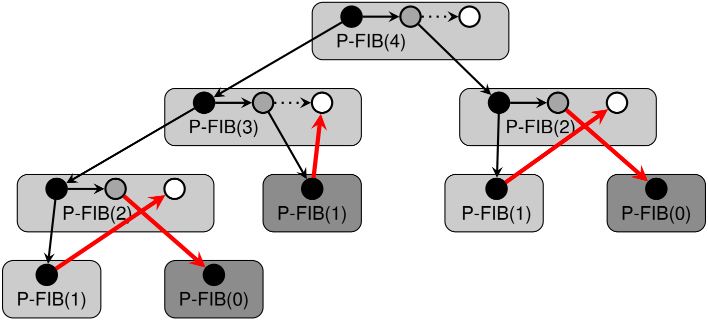


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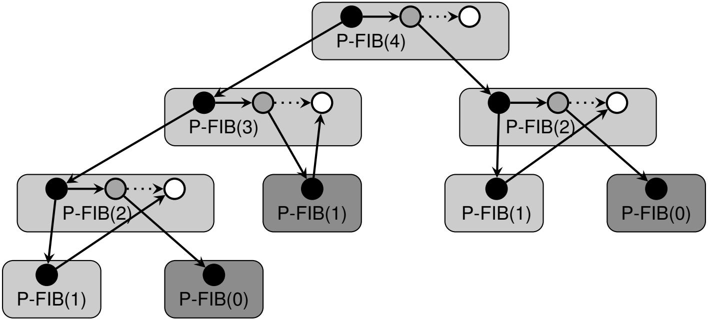


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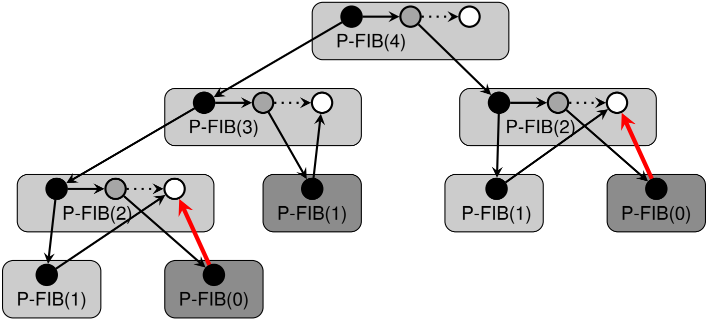


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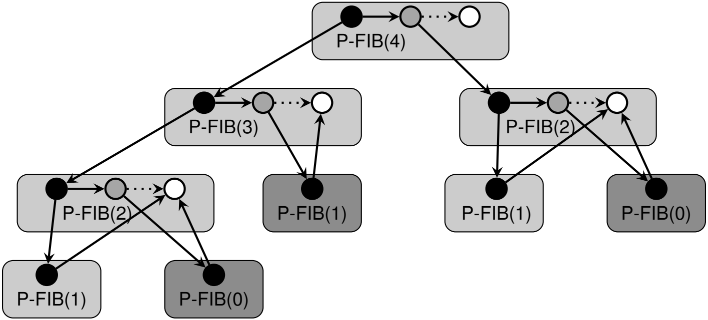


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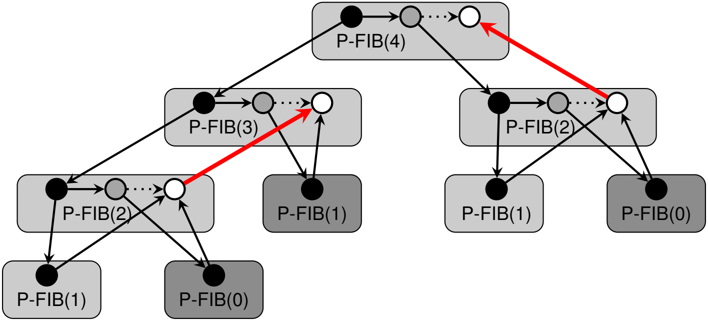


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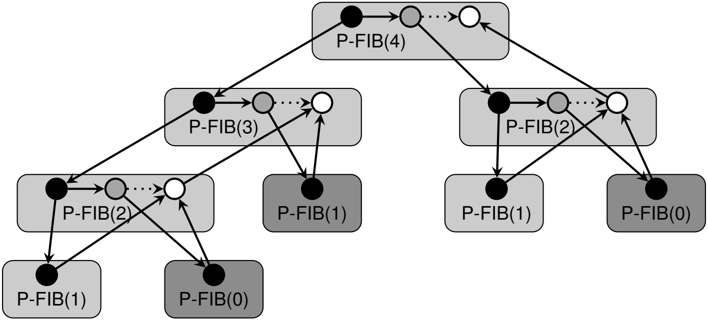


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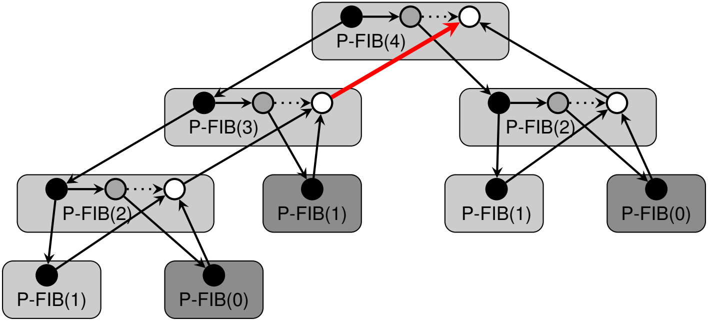


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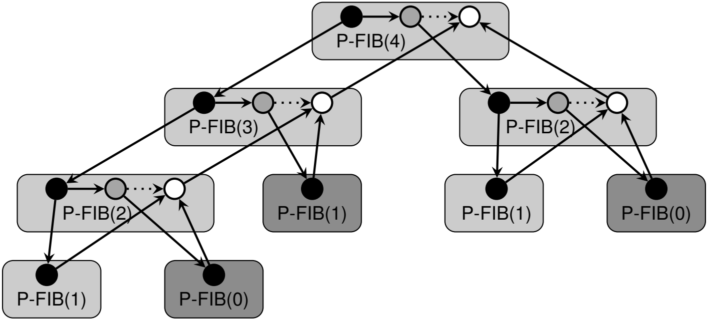


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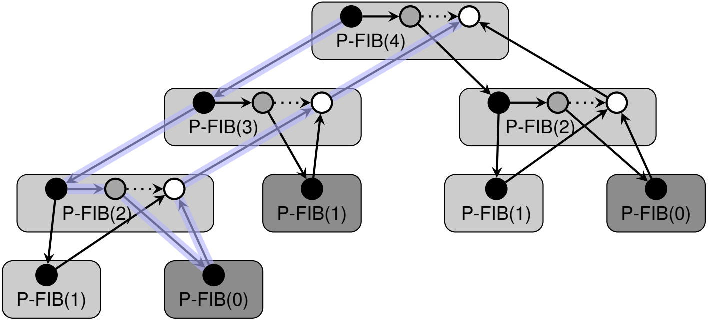


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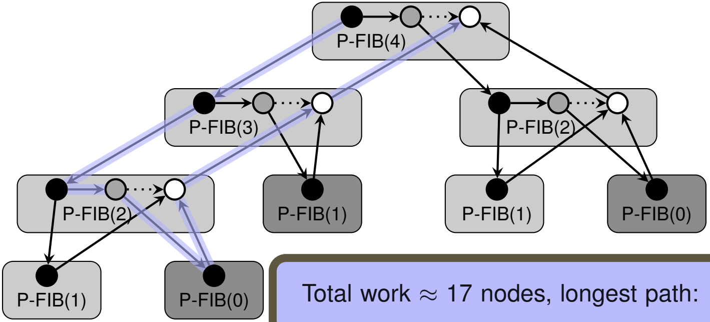

Total work $\approx$ 17 nodes, longest path: 8 nodes

```
P-FIB(n)
   if n<=1 return n
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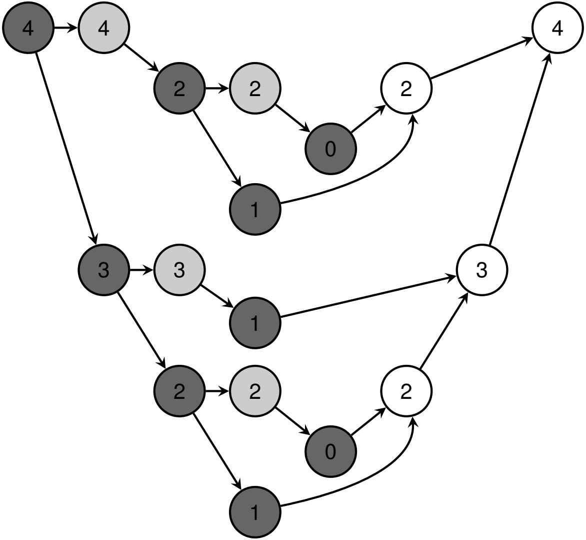
Computing Fibonacci Numbers in Parallel (DAG Perspective)



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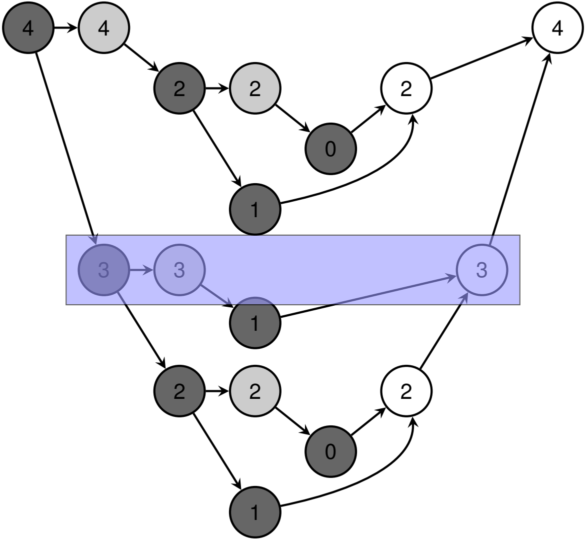

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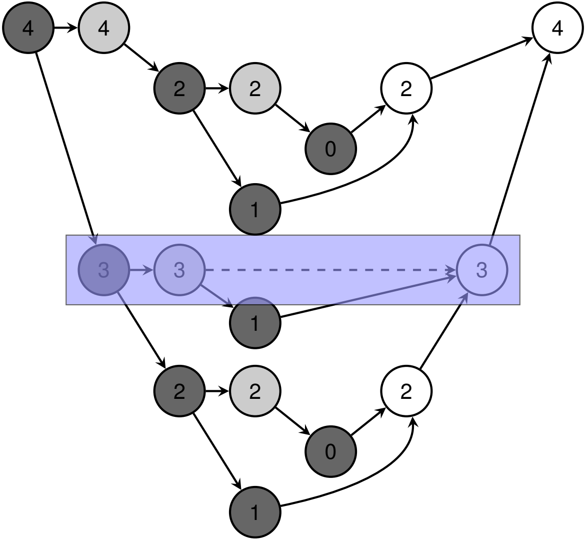

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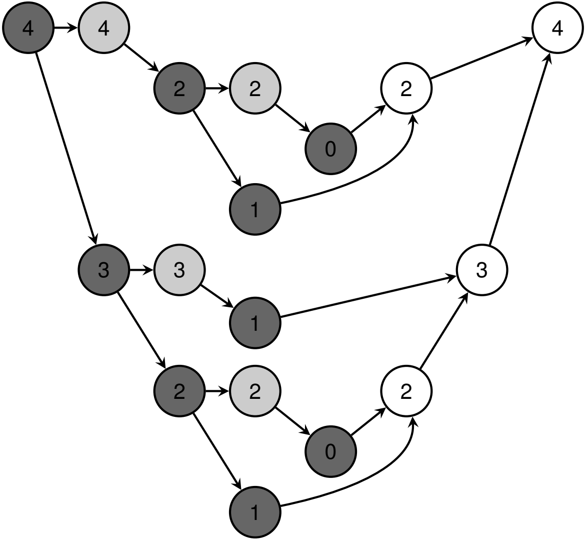

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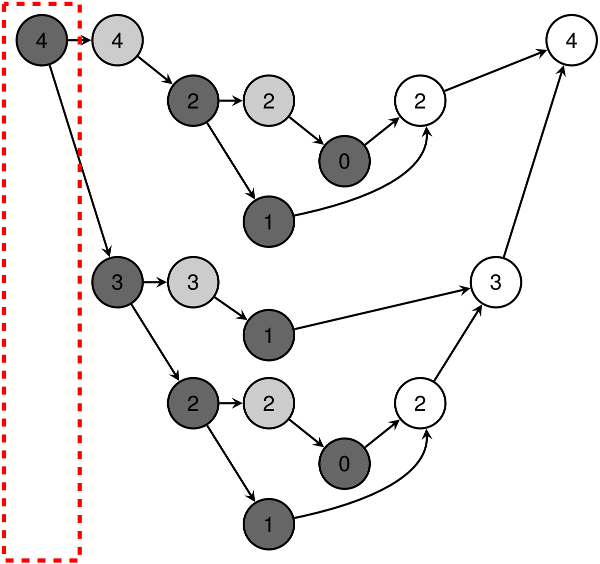

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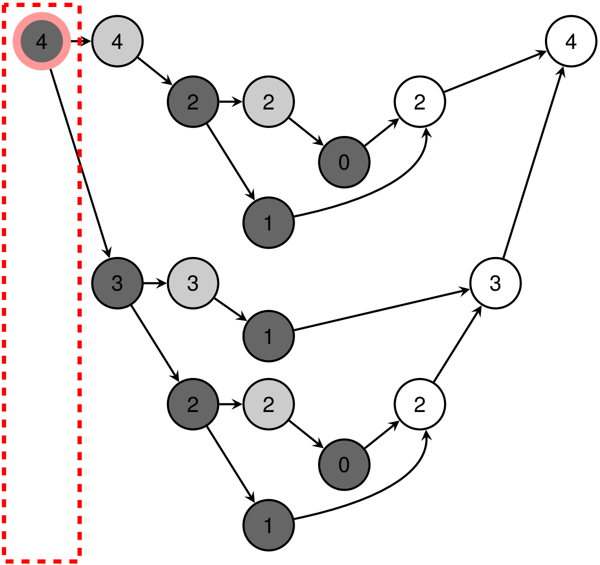

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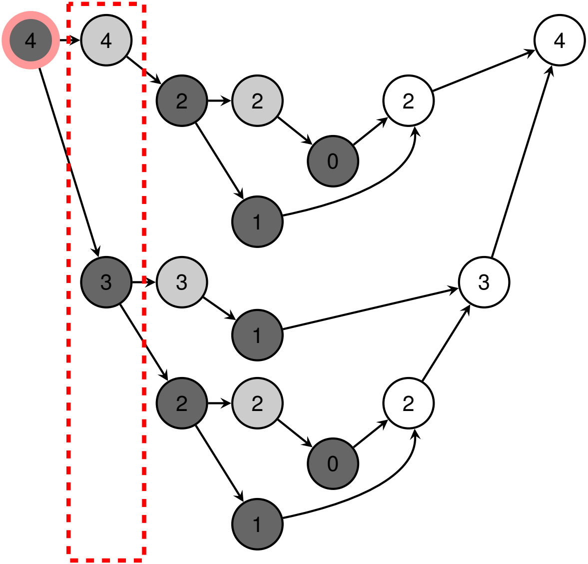

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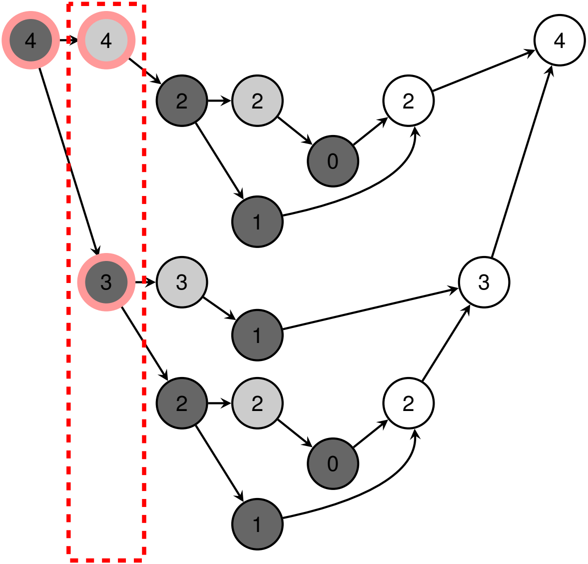

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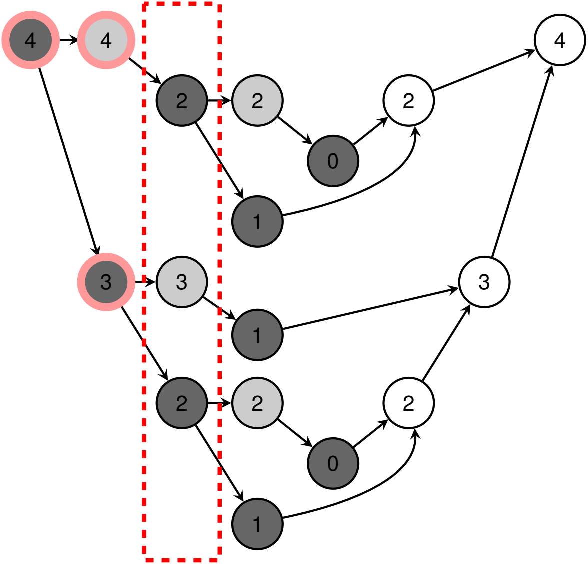
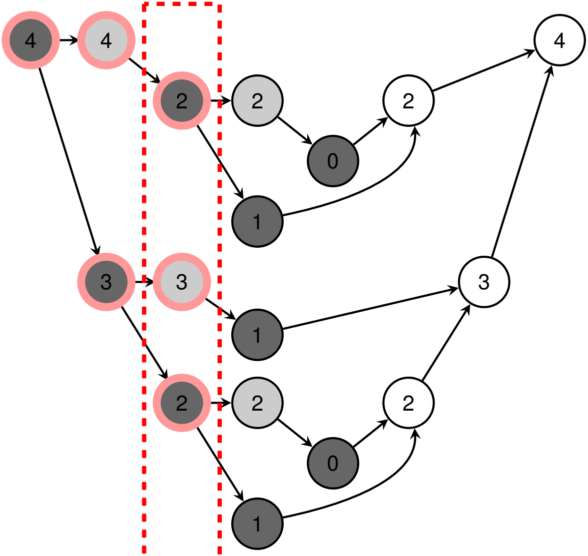


Computing Fibonacci Numbers in Parallel (DAG Perspective)



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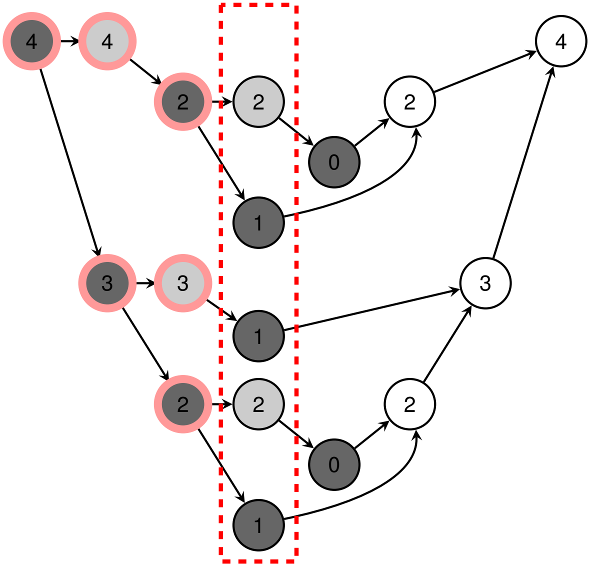

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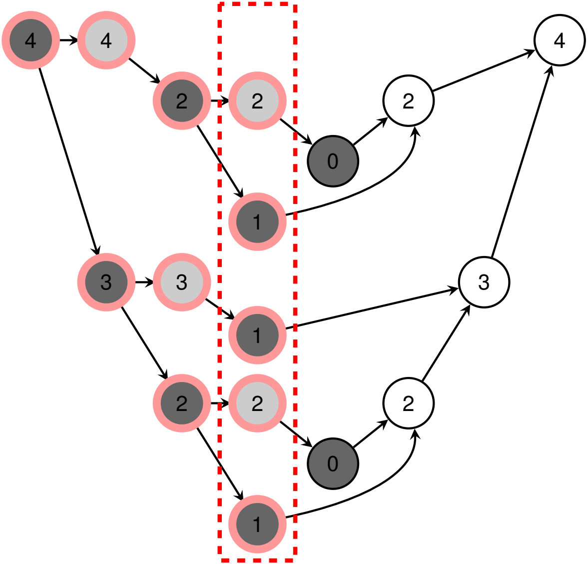

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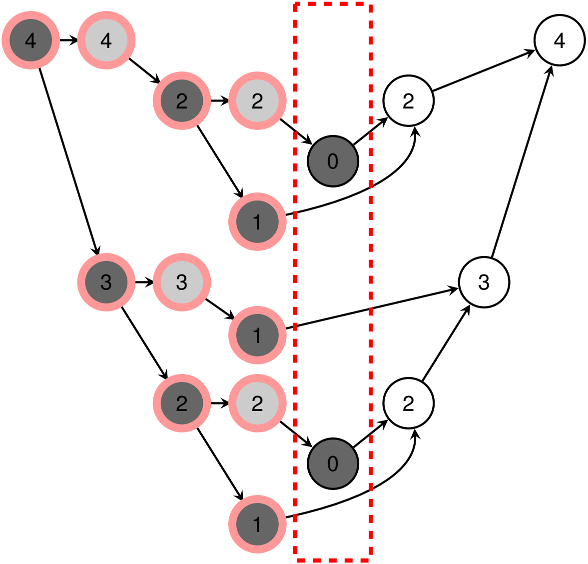

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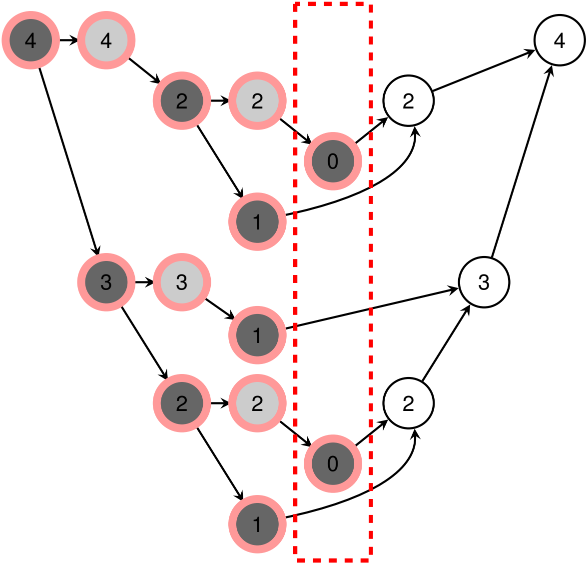

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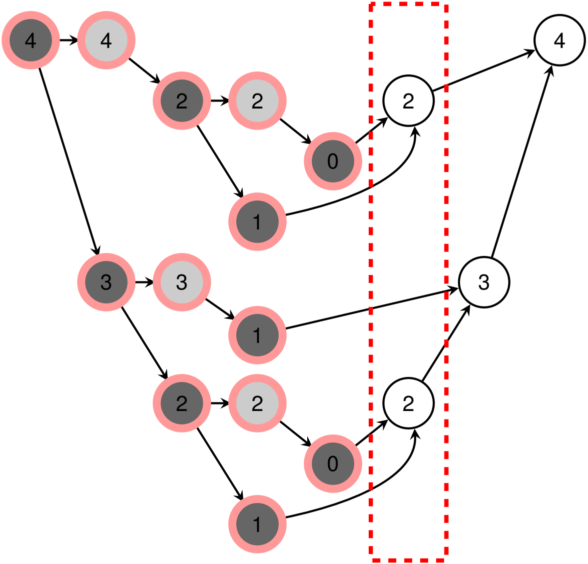
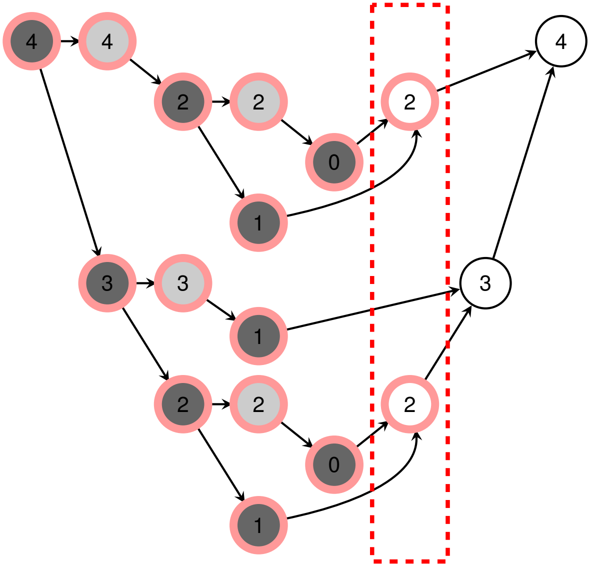


Computing Fibonacci Numbers in Parallel (DAG Perspective)



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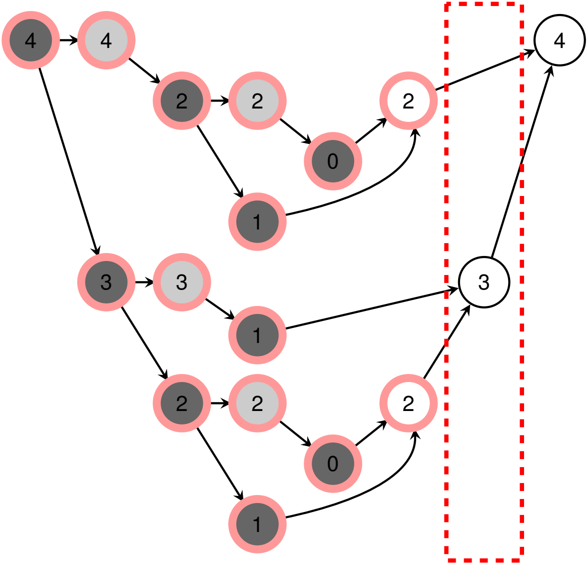

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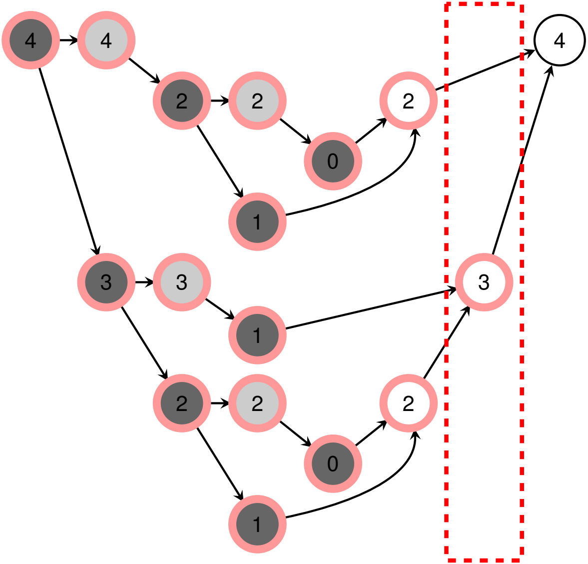

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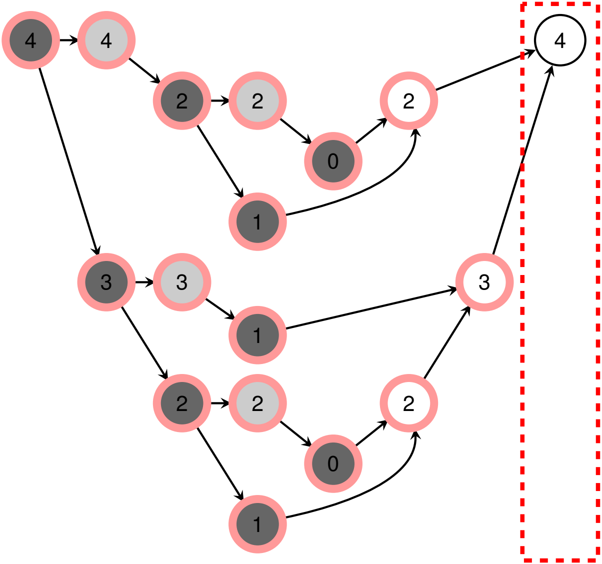

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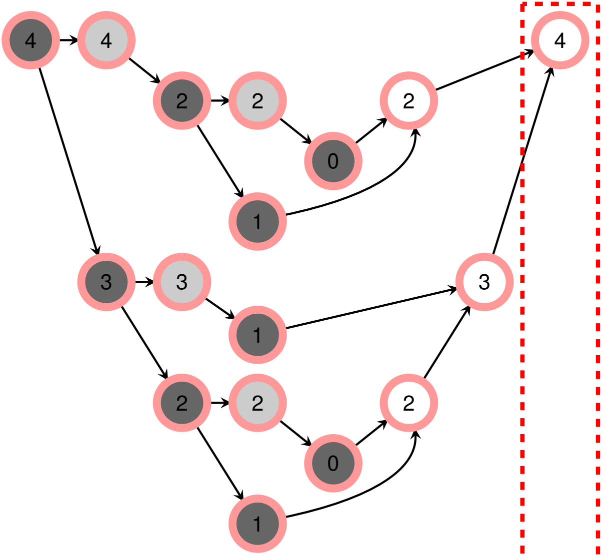

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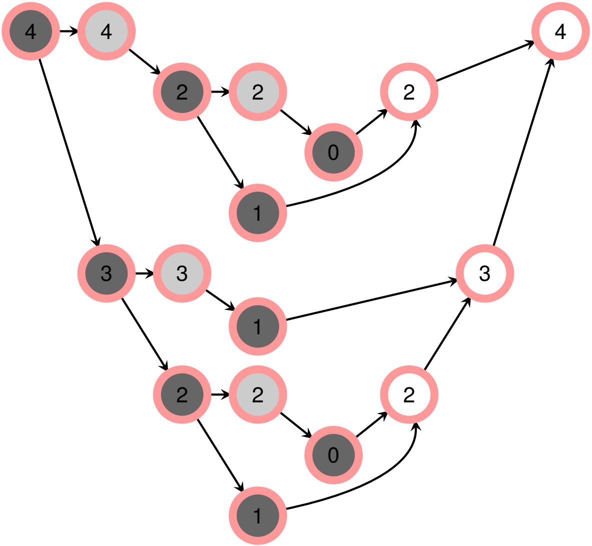

Performance Measures

Work

Total time to execute everything on a single processor.

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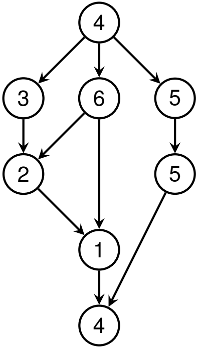
Total time to execute everything on a single processor.

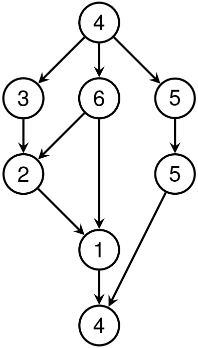


Performance Measures

$\sum = 30$

Work

Total time to execute everything on a single processor.



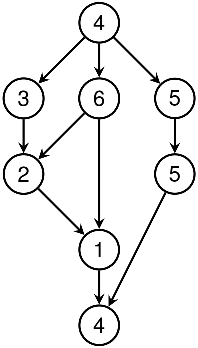
Performance Measures

Work

Total time to execute everything on a single processor.

Span

Longest time to execute the threads along any path.



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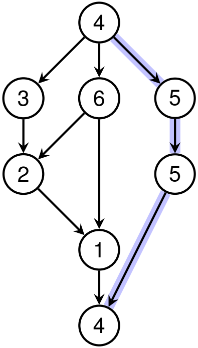


Performance Measures

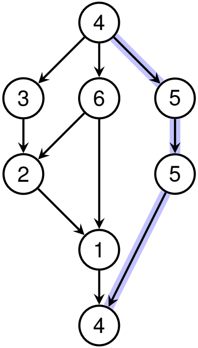
$\sum = 18$

Work

Total time to execute everything on a single processor.

Span

Longest time to execute the threads along any path.



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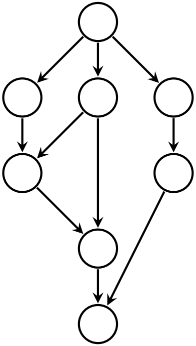


Performance Measures

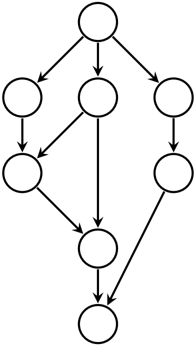
Work

Total time to execute everything on a single processor.

Span

Longest time to execute the threads along any path.

If each thread takes unit time, span is the length of the critical path.



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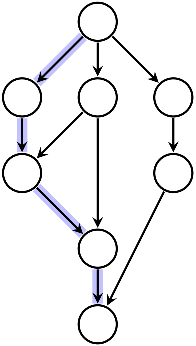

Performance Measures

Work

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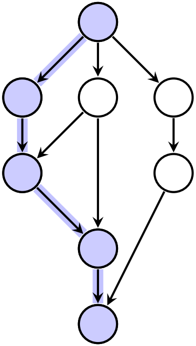
$\#\text{nodes} = 5$



Work Law and Span Law

Work Law and Span Law

- $T_1 =$ work, $T_\infty =$ span

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

Work Law and Span Law

- $T_1 =$ work, $T_\infty =$ span
- $P =$ number of (identical) processors
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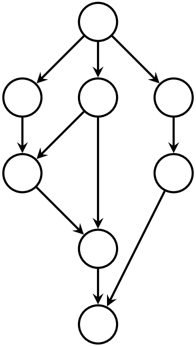
Running time actually also depends on scheduler etc.!

Work Law and Span Law

- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

Work Law

$$T_P \geq \frac{T_1}{P}$$



Work Law and Span Law

- T_1 = work, T_∞ = span
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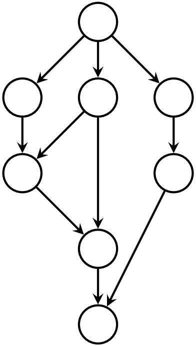
$T_1 = 8,\ P = 2$



Work Law

$$T_P \geq \frac{T_1}{P}$$

Time on P processors can't be shorter than if all work all time

Work Law and Span Law

- T_1 = work, T_∞ = span
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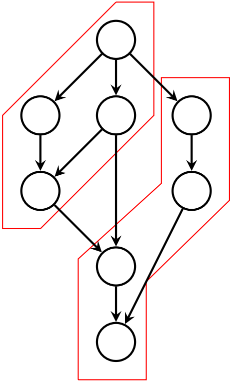
- T_P = running time on P processors

$T_1 = 8$, $P = 2$



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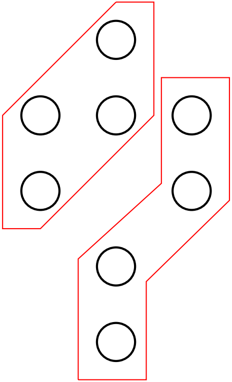

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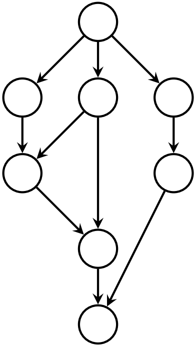


Work Law and Span Law

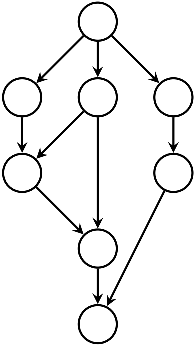
- T_1 = work, T_∞ = span
- P = number of (identical) processors
- T_P = running time on P processors

Work Law

$$T_P \geq \frac{T_1}{P}$$

Span Law

$$T_P \geq T_\infty$$



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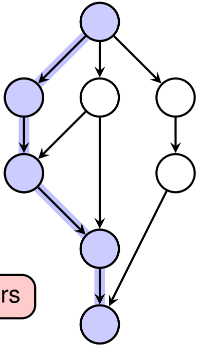

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Time on P processors can't be shorter than time on ∞ processors

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- Speed-Up: $\frac{T_1}{T_P}$

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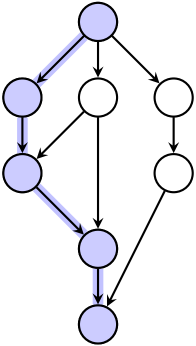

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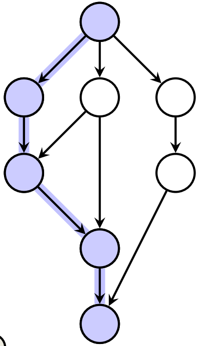

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- Speed-Up: $\frac{T_1}{T_P}$

Maximum Speed-Up bounded by P!

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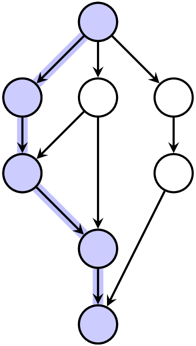

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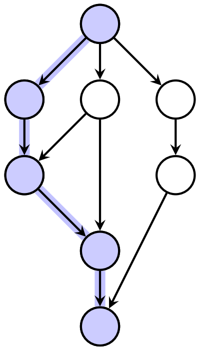

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Maximum Speed-Up for ∞ processors!

Outline

Introduction

Serial Matrix Multiplication

Digression: Multithreading

Multithreaded Matrix Multiplication

Warmup: Matrix Vector Multiplication

Remember: Multiplying an $n \times n$ matrix $A = (a_{ij})$ and n-vector $x = (x_j)$ yields an n-vector $y = (y_i)$ given by

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad \text{for } i = 1, 2, \ldots, n.$$

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The **parallel for**-loops can be used since different entries of y can be computed concurrently.

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How can a compiler implement the **parallel for**-loop?

Implementing `parallel for` based on Divide-and-Conquer

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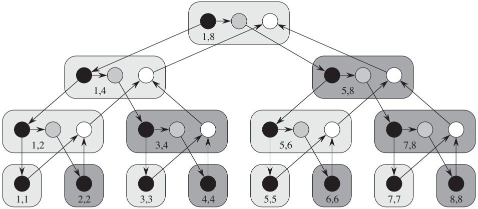

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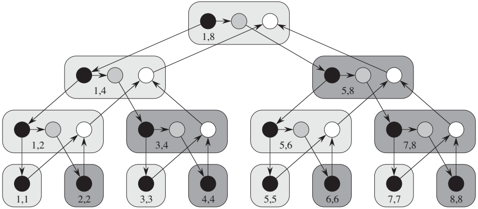

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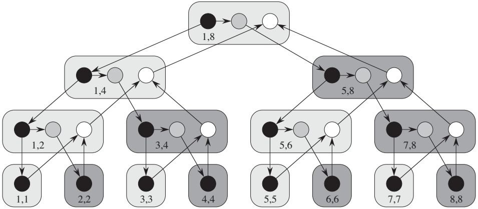

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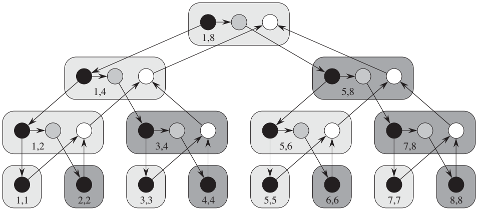

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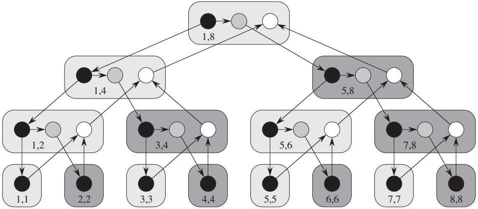

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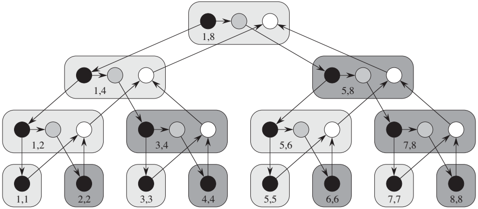


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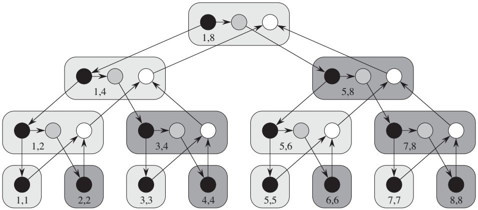

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Work is equal to running time of its serialization; overhead of recursive spawning does not change asymptotics.

$$T_\infty(n) = \Theta(\log n) + \max_{1 \leq i \leq n} \text{iter}(n)$$

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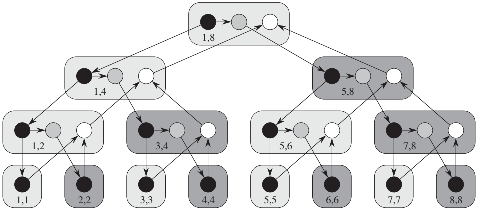

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$$T_\infty(n) = \Theta(\log n) + \max_{1 \leq i \leq n} \text{iter}(n)$$
$$= \Theta(n).$$

Span is the depth of recursive callings plus the maximum span of any of the n iterations.

Naive Algorithm in Parallel

```
P-SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
parallel for i = 1 to n
    parallel for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

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With a more careful implementation, $T_\infty(n) = O(\log n)$ (CLRS, Exercise 27.2-3)

P-SQUARE-MATRIX-MULTIPLY(A, B) has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) = \Theta(n)$.

The first two nested for-loops parallelise perfectly.

The Simple Divide&Conquer Approach in Parallel

```
P-MATRIX-MULTIPLY-RECURSIVE(C, A, B)
n = A.rows
if n == 1
    c11 = a11 b11
else let T be a new n × n matrix
    partition A, B, C, and T into n/2 × n/2 submatrices
        A11, A12, A21, A22; B11, B12, B21, B22; C11, C12, C21, C22;
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    sync
    parallel for i = 1 to n
        parallel for j = 1 to n
            cij = cij + tij
```

The Simple Divide&Conquer Approach in Parallel

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The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) =$

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$T_\infty(n) = T_\infty(n/2) + \Theta(\log n)$

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The same as before.

P-MATRIX-MULTIPLY-RECURSIVE has work $T_1(n) = \Theta(n^3)$ and span $T_\infty(n) = \Theta(\log^2 n)$.

$T_\infty(n) = T_\infty(n/2) + \Theta(\log n)$

Strassen's Algorithm in Parallel

Strassen's Algorithm (parallelised)

1. Partition each of the matrices into four $n/2 \times n/2$ submatrices

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   This step takes $\Theta(1)$ work and span by index calculations.

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   This step takes $\Theta(1)$ work and span by index calculations.

2. Create 10 matrices $S_1, S_2, \ldots, S_{10}$. Each is $n/2 \times n/2$ and is the sum or difference of two matrices created in the previous step.

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Matrix Multiplication and Matrix Inversion

Speedups for Matrix Inversion by an equivalence with Matrix Multiplication.

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If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

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If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

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$\Rightarrow$ We can compute AB in $O(I(n))$ time. $\square$

The Other Direction

Theorem 28.1 (Multiplication is no harder than Inversion)

If we can invert an $n \times n$ matrix in time $I(n)$, where $I(n) = \Omega(n^2)$ and $I(n)$ satisfies the regularity condition $I(3n) = O(I(n))$, then we can multiply two $n \times n$ matrices in time $O(I(n))$.

Theorem 28.2 (Inversion is no harder than Multiplication)

Suppose we can multiply two $n \times n$ real matrices in time $M(n)$ and $M(n)$ satisfies the two regularity conditions $M(n+k) = O(M(n))$ for any $0 \leq k \leq n$ and $M(n/2) \leq c \cdot M(n)$ for some constant $c < 1/2$. Then we can compute the inverse of any real nonsingular $n \times n$ matrix in time $O(M(n))$.

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Proof of this directon much harder (CLRS) – relies on properties of SPD matrices.

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Allows us to use Strassen's Algorithm to invert a matrix!

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