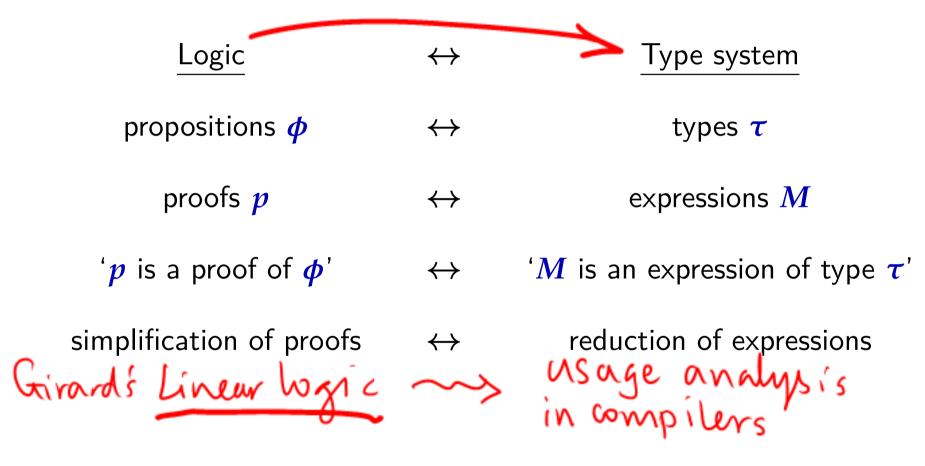
Logic	$\leftrightarrow$	Type system
propositions $\phi$	$\leftrightarrow$	types $ au$
proofs <i>p</i>	$\leftrightarrow$	expressions $M$
' $p$ is a proof of $\phi$ '	$\leftrightarrow$	' $M$ is an expression of type $ au$ '
simplification of proofs	$\leftrightarrow$	reduction of expressions





Linear implication -o T, 4+4 T+ 4-04

Γ+ φ-04

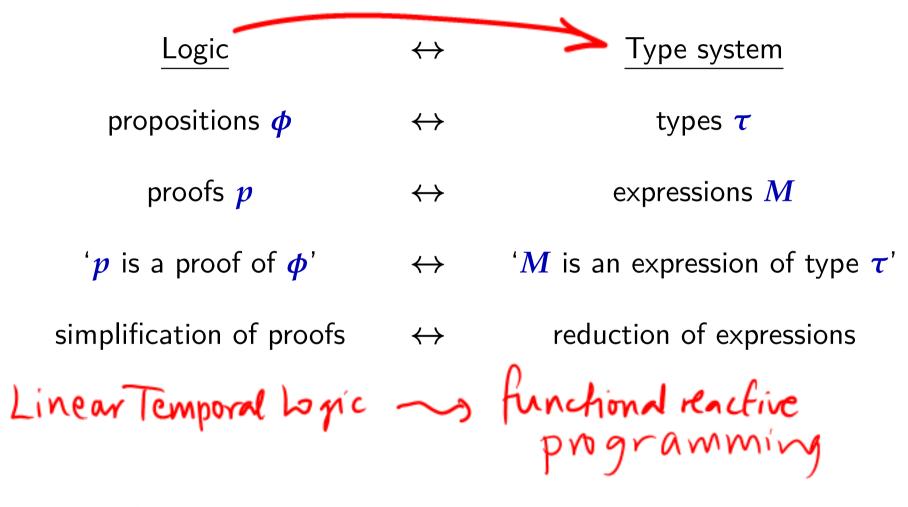
Γ-4-4 Γ-4-4 Γ-4-6)

Linear conjunction (tensor)

 $\frac{\Gamma + \varphi \quad \Delta + \psi}{\Gamma, \Delta + \varphi \otimes \psi} (\Gamma \cap \Delta = \emptyset)$ 

 $\frac{\Gamma + \Psi \otimes \Psi \quad \Delta, \Psi, \Psi + \Theta}{\Gamma, \Delta + \Theta} (\Gamma \wedge \Delta = \emptyset)$ 

## Applications

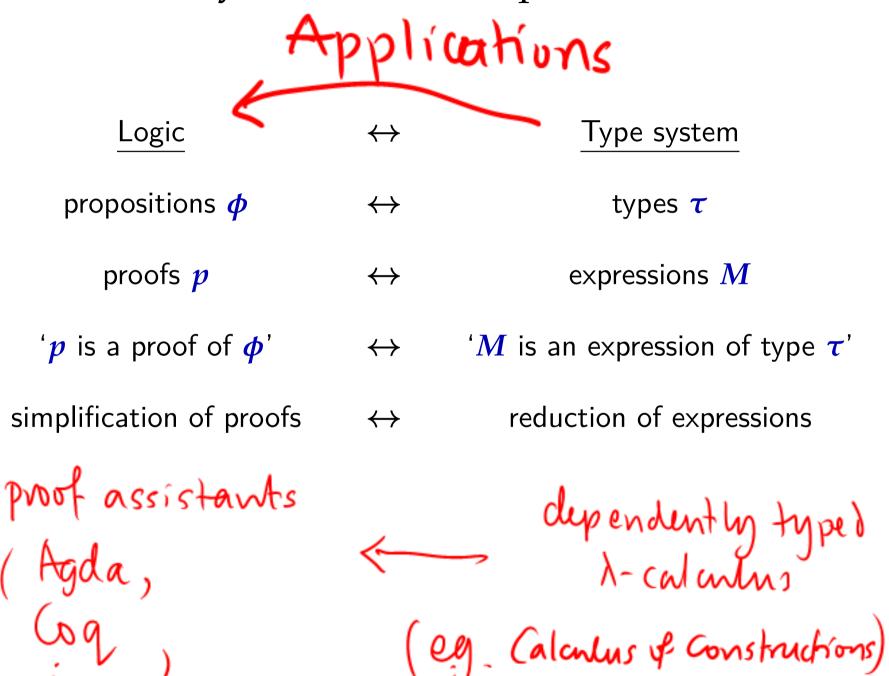


Model logics ~> partial evaluation & non-time code generation

### Type-inference versus proof search

```
Type-inference: given \Gamma and M, is there a type \tau such that \Gamma \vdash M : \tau? (For PLC/2IPC this is decidable.)
```

**Proof-search**: given  $\Gamma$  and  $\phi$ , is there a proof term M such that  $\Gamma \vdash M : \phi$ ? (For PLC/2IPC this is undecidable.)



Logic	$\leftrightarrow$	Type system
propositions $m{\phi}$	$\leftrightarrow$	types $ au$
proofs <i>p</i>	$\leftrightarrow$	expressions <i>M</i>
" $p$ is a proof of $\phi$ "	$\leftrightarrow$	' $M$ is an expression of type $ au$ '
simplification of proofs	$\leftrightarrow$	reduction of expressions
a logic of proposition		
	E.g.	
2IPC	$\leftrightarrow$	PLC
-Halso applied	10	predicate logi

higher-order intuitionistic

predicate

Togic

Togic

### Pure Type Systems – typing rules

(axiom) 
$$\rightarrow \vdash s_1 : s_2$$
 if  $\underline{(s_1, s_2)} \in \mathcal{A}$ 

(start) 
$$\frac{\Gamma \vdash A : s}{\Gamma_{\iota} x : A \vdash x : A}$$
 if  $x \notin dom(\Gamma)$ 

(weaken) 
$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma_{\iota} x : B \vdash M : A}$$
 if  $x \notin dom(\Gamma)$ 

$$(conv) \frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma \vdash M : B} \text{ if } A =_{\beta} B$$

$$(\operatorname{prod}) \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A (B) : s_3} \text{ if } \underline{(s_1, s_2, s_3) \in \mathcal{R}}$$

(abs) 
$$\frac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash \Pi x : A (B) : s}{\Gamma \vdash \lambda x : A (M) : \Pi x : A (B)}$$

$$(app) \frac{\Gamma \vdash M : \Pi x : A(B) \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B[N/x]}$$

(A, B, M, N) range over pseudoterms,  $s, s_1, s_2, s_3$  over sort symbols)

is the Pure Type System  $\lambda C$ , where  $C = (S_C, A_C, \mathcal{R}_C)$  is the PTS specification with

```
\mathcal{S}_{\mathbf{C}} \triangleq \{ \mathsf{Prop}, \mathsf{Set} \} (Prop = a sort of propositions, Set = a sort of types) \mathcal{A}_{\mathbf{C}} \triangleq \{ (\mathsf{Prop}, \mathsf{Set}) \} (Prop is one of the types) \mathcal{R}_{\mathbf{C}} \triangleq \{ (\mathsf{Prop}, \mathsf{Prop}, \mathsf{Prop}), (\mathsf{Set}, \mathsf{Prop}, \mathsf{Prop}), (\mathsf{Prop}, \mathsf{Set}, \mathsf{Set}), (\mathsf{Set}, \mathsf{Set}, \mathsf{Set}) \}
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(\text{Prop, Set, Set}), (\text{Set, Set, Set})\}
1. Prop has implications, \phi \rightarrow \psi = \Pi x : \phi(\psi) (where \phi, \psi : \text{Prop and } x \notin fv(q)).

2. Prop has universal quantifications over elements of a type, \Pi x : A(\phi(x)) (where A : \text{Set and } x : A \vdash \phi(x) : \text{Prop}).
```

N.B. A might be Prop  $(\lambda 2 \subseteq \lambda C)$ .

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4. Set has dependent function types,  $\Pi x : A(B(x))$  (where A : Set and  $x : A \vdash B(x) : Set$ ).

 $\Pi x : p(A(x))$  (where  $p : Prop and <math>x : p \vdash A(x) : Set$ ).

### Some general properties of $\lambda C$

▶ It extends both  $\lambda 2$  (PLC) and  $\lambda \omega$  ( $F_{\omega}$ ).

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- ► Type-checking and typeability are decidable.

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- ▶ It extends both  $\lambda 2$  (PLC) and  $\lambda \omega$  ( $F_{\omega}$ ).
- $\triangleright$   $\lambda$ **C** is strongly normalizing.
- Type-checking and typeability are decidable.
- ▶  $\lambda C$  is logically consistent (relative to the usual foundations of classical mathematics), that is, there is no pseudo-term t satisfying  $\Diamond \vdash t : \Pi p : \text{Prop}(p)$ .

Indeed there is no proof of LEM  $(\Pi p : \text{Prop}(\neg p \lor p))$ .

# Logical operations definable in 2IPC



- ► Truth  $\top \triangleq \forall p (p \rightarrow p)$
- ► Falsity  $\bot \triangleq \forall p (p)$
- ► Conjunction  $\phi \land \psi \triangleq \forall p ((\phi \rightarrow \psi \rightarrow p) \rightarrow p)$  (where  $p \notin fv(\phi, \psi)$ )
- **▶** *Disjunction*  $\phi \lor \psi \triangleq \forall p ((\phi \rightarrow p) \rightarrow (\psi \rightarrow p) \rightarrow p)$  (where  $p \notin fv(\phi, \psi)$ )
- ▶ Negation  $\neg \phi \triangleq \phi \rightarrow \bot$
- ▶ Bi-implication  $\phi \leftrightarrow \psi \triangleq (\phi \rightarrow \psi) \land (\psi \rightarrow \phi)$

$$P \rightarrow q \stackrel{\triangle}{=} TT x: P(q) \quad x \notin fr(p)$$
  
 $\forall P(\varphi) \stackrel{\triangle}{=} TT p: Prop(\varphi)$ 

#### Leibniz equality in $\lambda C$

Gottfried Wilhelm Leibniz (1646–1716), identity of indiscernibles:

duo quaedam communes proprietates eorum nequaquam possit (two distinct things cannot have all their properties in common).

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Given  $\Gamma \vdash A : Set \text{ in } \lambda C$ , we can define

$$ext{Eq}_A riangleq \lambda x, y : A\left(\Pi P : A o ext{Prop}\left(P \, x \leftrightarrow P \, y
ight)
ight)$$

satisfying  $\Gamma \vdash \text{Eq}_A : A \to A \to \text{Prop}$  and giving a well-behaved (but not extensional) equality predicate for elements of type A.

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$$p \Leftrightarrow q \stackrel{\Delta}{=} (p \Rightarrow q) \wedge (q \Rightarrow p)$$

#### **Functional extensionality:**

$$ext{FunExt}_{A,B} riangleq \Pi f,g:A o B \ ( \Pi x:A \left( ext{Eq}_{B} \left( f \, x 
ight) \left( g \, x 
ight) 
ight) o ext{Eq}_{A o B} \, f \, g 
ight)$$

#### **Functional extensionality:**

```
	ext{FunExt}_{A,B} 	riangleq \Pi f,g:A	o B \ ( \Pi x:A \left( 	ext{Eq}_B \left( f \, x 
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```

If  $\Gamma \vdash A, B$ : Set in  $\lambda C$ , then  $\Gamma \vdash \operatorname{Ext}_{A,B}$ : Prop is derivable, but for some A,B there does not exist a pseudo-term t for which  $\Gamma \vdash t : \operatorname{Ext}_{A,B}$  is derivable.

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#### Propositional extensionality:

$$exttt{PropExt} riangleq oldsymbol{\Pi} p, q: exttt{Prop} \left( (p \leftrightarrow q) 
ightarrow exttt{Eq}_{ exttt{Prop}} \, p \, q 
ight)$$

 $\diamond \vdash \text{PropExt} : \text{Prop}$  is derivable in  $\lambda C$ , but there does not exist a pseudo-term t for which  $\diamond \vdash t : \text{PropExt}$  is derivable.

This is a weak form of Voevodsky's Univalence Axiom - unvently a Hot topic in type theory research Propositional extensionality: (Homotopy Type Theory)

ig> PropExt  $riangleq \Pi p, q$  : Prop  $((p \leftrightarrow q) 
ightarrow ext{Eq}_{ ext{Prop}} \, p \, q)$ 

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