Mathematical Methods for Computer Science

UNIVERSITY OF CAMBRIDGE

Computer Laboratory

Computer Science Tripos, Part IB

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Exercise problems –

Fourier and related methods

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1. Using the Euclidean norm on an inner product space $V = \mathbb{R}^3$, for the following vectors $u, v \in V$ whose span is a linear subspace of V,

$$u = \left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}} \right)$$

$$v = \left(\sqrt{3}, \ -\frac{\sqrt{3}}{2}, \ -\frac{\sqrt{3}}{2} \right)$$

demonstrate both whether u, v form an *orthogonal system,* and also whether they form an *orthonormal system.*

2. (a) Define linear independence and linear dependence for the set of vectors $\{v_1, v_2, \ldots, v_n\}$ of a vector space V over a field $\mathbb{F}$ of scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$.

   (b) For vectors $u, v \in V$ in linear space $V = \mathbb{R}^n$ with $u = (u_1, u_2, \ldots, u_n)$, define the Euclidean norm $||u||$, and state the triangle inequality for $||u + v||$.

3. Let V be an inner product space spanned by an orthonormal system of vectors $\{e_1, e_2, \ldots, e_n\}$ so that $\forall i \neq j$ their inner product $\langle e_i, e_j \rangle = 0$, but every e_i is a unit vector so $\langle e_i, e_i \rangle = 1$. We wish to represent a data set consisting of vectors $u \in \text{span}\{e_1, e_2, \ldots, e_n\}$ in this space as a linear combination of the orthonormal vectors: $u = \sum_{i=1}^{n} a_i e_i$. Derive how the coefficients a_i can be determined for any vector u, and comment on the computational advantage of representing the data in an orthonormal system.

4. An inner product space E containing piecewise continuous complex functions $f(x)$ and $g(x)$ on some interval is spanned by the orthonormal basis functions $\{e_i\}$ used in the Fourier series. Thus complex coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ exist such that $f(x) = \sum_i \alpha_i e_i(x)$ and $g(x) = \sum_i \beta_i e_i(x)$.

   (a) Show that $\langle f, g \rangle = \sum_i \alpha_i \overline{\beta_i}$.

   (b) Would the same result hold if the orthonormal basis functions $\{e_i\}$ that span E were *not* the Fourier basis? Justify your answer, and provide the name for coefficients $\{\alpha_i\}$ and $\{\beta_i\}$ in such a case.

5. Using complex exponentials, prove the following trigonometric identity, which describes the multiplicative modulation of one cosine wave by another as being simply the sum of a different pair of cosine waves:

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a+b)x) + \frac{1}{2}\cos((a-b)x)$$

6. Calculate the Fourier series of the function $f(x)$ $(x \in [-\pi, \pi])$ defined by

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 0 & -\pi \le x < 0\,. \end{cases}$$

Find also the complex Fourier series for $f(x)$.

7. From the well-known fact that a periodic squarewave ($f(x) = 1$ for $0 < x < \pi$, $f(x) = -1$ for $\pi < x < 2\pi$, $\cdots$) has the following Fourier series

$$f(x) = \frac{4}{\pi}\left[\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \cdots\right],$$

produce the first four terms of the Fourier series for the triangle-wave whose derivative is this squarewave. Comment on the relative rates of convergence of these two series, and state the general rule about series convergence rates for periodic functions that become impulsive first in their n^{th} derivative.

8. Suppose that $f(x)$ is a 2π-periodic function with complex Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}\,.$$

Now consider the shifted version of $f(x)$ given by

$$g(x) = f(x - x_0)$$

where x_0 is a constant. Find the relationship between the complex Fourier coefficients of $g(x)$ in terms of those of $f(x)$. How do the magnitudes of the corresponding coefficients compare?

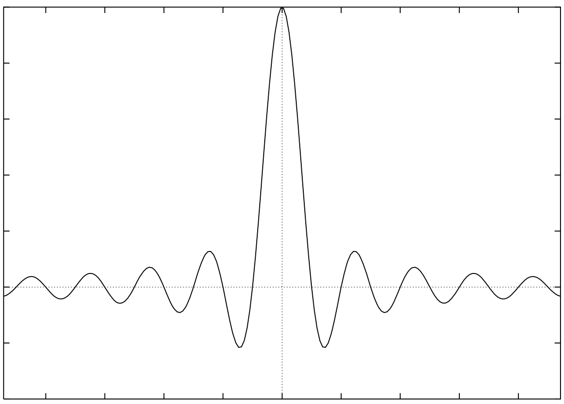
9. The Modulation Theorem asserts that if $f(x)$ has Fourier transform $F(\omega)$, then modulating $f(x)$ at frequency c (multiplying it by e^{icx}) simply shifts its transform up by c to become $F(\omega - c)$. Prove this, and explain one important practical application of this property.

10. For a function $f(x)$ whose Fourier Transform is $F(\omega)$, what is the Fourier Transform of $f^{(n)}(x)$, the n^{th} derivative of $f(x)$ with respect to x? Explain how Fourier methods make it possible to define non-integer orders of derivatives, and name one scientific field in which it is useful to take half-order derivatives.

11. Show how Fourier methods facilitate solution of differential equations such as the following, in which the non-zero function $g(x)$ is known, its Fourier Transform $G(\omega)$ can be computed, and a, b, c are constant coefficients. Derive an expression for $f(x)$ that is a solution to this differential equation, assuming its Fourier Transform exists.

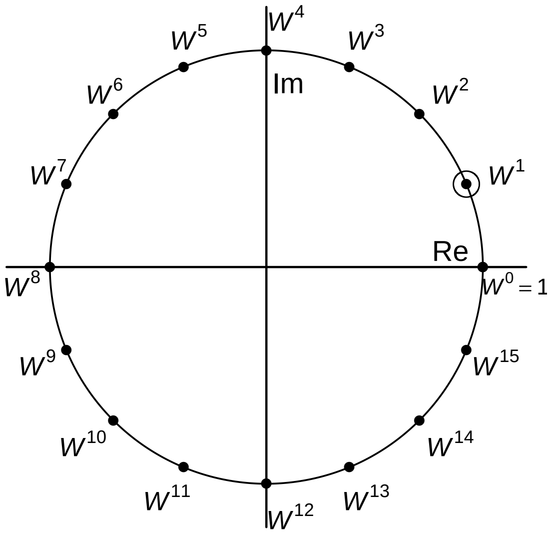
$$a\frac{d^2f(x)}{dx^2} + b\frac{df(x)}{dx} + cf(x) = g(x)$$

12. The function $\text{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$ for $x \neq 0$ as plotted here plays an important role in the Sampling Theorem. By considering its Fourier Transform, show that this function is unchanged in form after convolution with itself, and show that it even remains unchanged in form after convolution with any higher frequency sinc function $\text{sinc}(ax)$ for $a > 1$, but that if $0 < a < 1$, then the result is instead that lower frequency sinc function $\text{sinc}(ax)$.



13. Using a diagram in the complex plane showing the N^{th} roots of unity, explain why all the values of complex exponentials that are needed for computing the Discrete Fourier Transform of N data points are powers of a primitive N^{th} root of unity (circled below for $N = 16$), and explain why such factorisation greatly reduces the number of multiplications required in a Fast Fourier Transform.

14. Consider a sequence $f[n]$ $(n = 0, 1, \ldots, 15)$ with Fourier coefficients $F[k]$ $(k = 0, 1, \ldots, 15)$. Using the 16^{th} roots of unity as labelled around the unit circle in powers of w^1, the primitive 16^{th} root of unity, construct a sequence of these 16 roots w^i that could be used to compute $F[3]$.



15. What sets of frequencies are required to perform the following analyses?
    - Fourier transform of a non-periodic continuous function;
    - Fourier analysis of a piecewise continuous periodic function with period 2π;
    - Wavelet transform of a non-periodic function, either continuous or discrete.

    Comment on the relationship between the density of frequencies required and the role of "locality" in the analysis.

16. Show how a generating (or "mother") wavelet $\Psi(x)$ can spawn a family of "daughter" wavelets $\Psi_{jk}(x)$ by simple shifting and scaling ("dyadic") operations, and explain the advantages of representing continuous functions in terms of such a family of self-similar dilates and translates of a mother wavelet.