L11: Algebraic Path Problems with applications to Internet Routing Lectures 01 — 08

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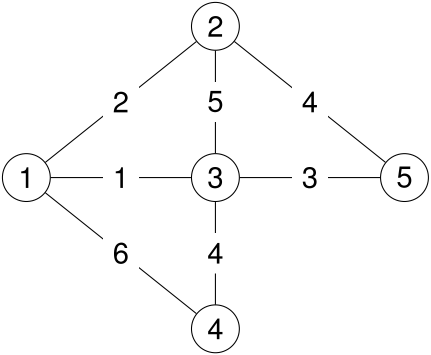
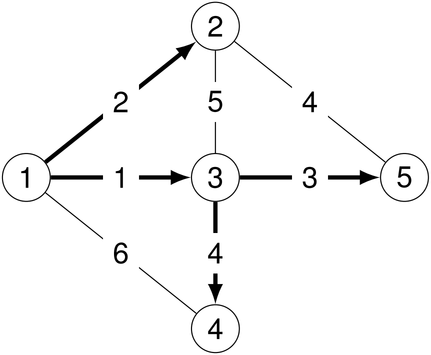
Shortest paths example, $\mathrm{sp} = (\mathbb{N}^{\infty},\ \min,\ +, \infty,\ 0)$



The adjacency matrix

$$
\mathbf{A} \;=\; \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
\infty & 2 & 1 & 6 & \infty \\
2 & \infty & 5 & \infty & 4 \\
1 & 5 & \infty & 4 & 3 \\
6 & \infty & 4 & \infty & \infty \\
\infty & 4 & 3 & \infty & \infty
\end{array} \right]
\end{array}
$$

Shortest paths solution



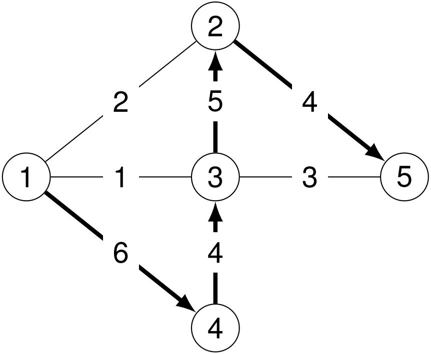
$$\mathbf{A}^* \quad = \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

solves this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from i to j.

Widest paths example, $\mathrm{bw} = (\mathbb{N}^{\infty}, \max, \min, 0, \infty)$



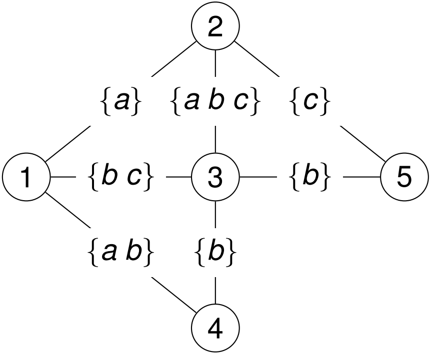
$$\mathbf{A}^* = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \infty & 4 & 4 & 6 & 4 \\ 4 & \infty & 5 & 4 & 4 \\ 4 & 5 & \infty & 4 & 4 \\ 6 & 4 & 4 & \infty & 4 \\ 4 & 4 & 4 & 4 & \infty \end{bmatrix} \end{array}$$

solves this global optimality problem:

$$\mathbf{A}^*(i, j) = \max_{p \in P(i, j)} w(p),$$

where $w(p)$ is now the minimal edge weight in p.

Unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cup,\ \cap, \{\},\ \{a,\ b,\ c\})$



We want $\mathbf{A}^*$ to solve this global optimality problem:

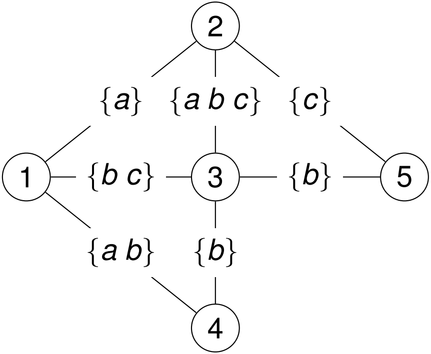
$$\mathbf{A}^*(i,\ j) = \bigcup_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the intersection of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{A}^*(i,\ j)$ to mean that there is at least one path from i to j with x in every arc weight along the path.

$$\mathbf{A}^*(4,\ 1) = \{a,\ b\} \qquad \mathbf{A}^*(4,\ 5) = \{b\}$$

Another unfamiliar example, $(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$



We want matrix **R** to solve this global optimality problem:

$$\mathbf{A}^*(i,\ j) = \bigcap_{p \in P(i,\ j)} w(p),$$

where $w(p)$ is now the union of all edge weights in p.

For $x \in \{a,\ b,\ c\}$, interpret $x \in \mathbf{R}(i,\ j)$ to mean that every path from i to j has at least one arc with weight containing x.

$$\mathbf{A}^*(4,\ 1) = \{b\} \quad \mathbf{A}^*(4,\ 5) = \{b\} \quad \mathbf{A}^*(5,\ 1) = \{\}$$

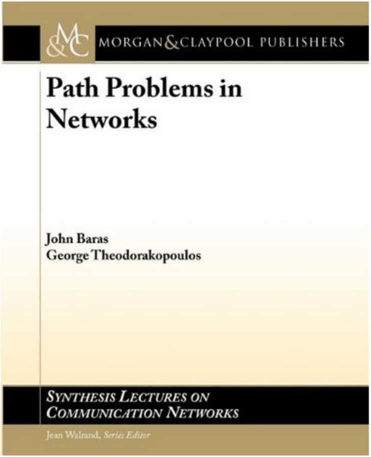
We will start by looking at Semirings

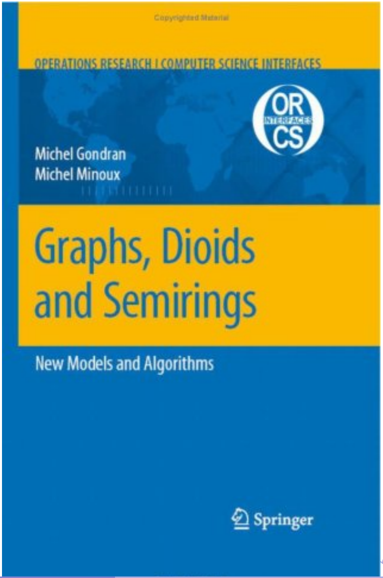
name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$	possible routing use
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0	minimum-weight routing
bw	$\mathbb{N}^{\infty}$	max	min	0	∞	greatest-capacity routing
rel	$[0, 1]$	max	$\times$	0	1	most-reliable routing
use	$\{0, 1\}$	max	min	0	1	usable-path routing
	2^W	$\cup$	$\cap$	$\{\}$	W	shared link attributes?
	2^W	$\cap$	$\cup$	W	$\{\}$	shared path attributes?

A wee bit of notation!

Symbol	Interpretation
$\mathbb{N}$	Natural numbers (starting with zero)
$\mathbb{N}^{\infty}$	Natural numbers, plus infinity
$\overline{0}$	Identity for $\oplus$
$\overline{1}$	Identity for $\otimes$

Recommended Reading on Semiring Theory





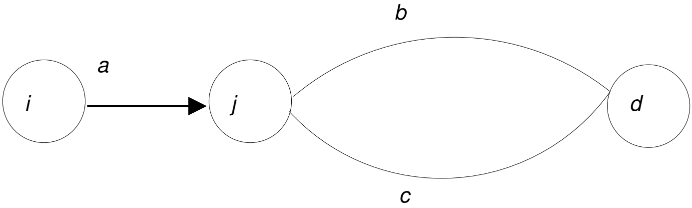
Semirings (generalise $(\mathbb{R}, +, \times, 0, 1)$)

We will look at the axioms of semirings. The most important are

distributivity

$$\begin{array}{lclcl} \lessdot & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\ \diagdown & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c) \end{array}$$

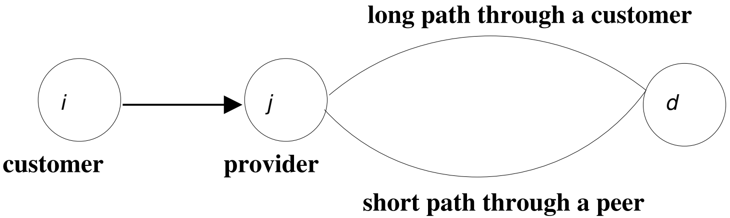
Distributivity, illustrated



$$a \otimes (b \oplus c) \quad = \quad (a \otimes b) \oplus (a \otimes c)$$

j makes the choice $=$ *i* makes the choice

Should distributivity hold in Internet Routing? No!



- *j* prefers long path though one of its customers (not the shorter path through a competitor)
- given two routes from a provider, *i* prefers the one with a shorter path

More on this later in the term ...

The (Tentative) Plan

1	9 October	:	Motivation, overveiw
2	13 October	:	Semigroups and Orders
3	16 October	:	Semirings — Theory, algorithms
4	20 October	:	Semirings — Constructions
5	23 October	:	Semirings — Constructions
6	27 October	:	Semirings — Constructions
7	30 October	:	Beyond Semirings — AMEs — "functions on arc
8	3 November	:	AME Constructions (**HW 1 due**)
9	6 November	:	Protocols : RIP, EIGRP (from a theoretical persp
10	10 November	:	Inter-domain routing in the Internet I
11	13 November	:	Inter-domain routing in the Internet II
12	17 November	:	Beyond Semirings — Global vs Local optimality
13	20 November	:	More on Global vs Local optimality
14	24 November	:	Dijkstra revisited (**HW 2 due**)
15	27 November	:	Bellman-Ford revisited
16	1 December	:	Other algorithms
	12 January	:	**HW 3 due**

- What is a semigroup?
- A few important semigroup properties.
- Cayley's Theorem for semigroups.
- Constructing new semigroups from old.
- Homework 1.

Semigroups

Semigroup

A semigroup $(S, \bullet)$ is a non-empty set S with a binary operation such that

$$\mathbb{AS} \quad \text{associative} \quad \equiv \quad \forall a, b, c \in S, \; a \bullet (b \bullet c) = (a \bullet b) \bullet c$$

Important Assumption — We will ignore trival semigroups

We will impicitly assume that $2 \leq | S |$.

Note

Many useful binary operations are not semigroup operations. For example, $(\mathbb{R}, \bullet)$, where $a \bullet b \equiv (a + b)/2$.

Some Important Semigroup Properties

$$
\begin{array}{ccll}
\mathbb{ID} & \text{identity} & \equiv & \exists \alpha \in S,\ \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha \\
\mathbb{AN} & \text{annihilator} & \equiv & \exists \omega \in S,\ \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega \\
\mathbb{CM} & \text{commutative} & \equiv & \forall a, b \in S,\ a \bullet b = b \bullet a \\
\mathbb{SL} & \text{selective} & \equiv & \forall a, b \in S,\ a \bullet b \in \{a,\ b\} \\
\mathbb{IP} & \text{idempotent} & \equiv & \forall a \in S,\ a \bullet a = a
\end{array}
$$

A semigroup with an identity is called a monoid.

Note that

$$\mathbb{SL}(S,\ \bullet) \implies \mathbb{IP}(S,\ \bullet)$$

A few concrete semigroups

S	$\bullet$	description	α	ω	$\mathbb{CM}$	$\mathbb{SL}$	$\mathbb{IP}$
S	left	x left $y = x$				$\star$	$\star$
S	right	x right $y = y$				$\star$	$\star$
S^*	$\cdot$	concatenation	ϵ				
S^+	$\cdot$	concatenation					
$\{t,\ f\}$	$\wedge$	conjunction	t	f	$\star$	$\star$	$\star$
$\{t,\ f\}$	$\vee$	disjunction	f	t	$\star$	$\star$	$\star$
$\mathbb{N}$	min	minimum		0	$\star$	$\star$	$\star$
$\mathbb{N}$	max	maximum	0		$\star$	$\star$	$\star$
2^W	$\cup$	union	$\{\}$	W	$\star$		$\star$
2^W	$\cap$	intersection	W	$\{\}$	$\star$		$\star$
$\text{fin}(2^U)$	$\cup$	union	$\{\}$		$\star$		$\star$
$\text{fin}(2^U)$	$\cap$	intersection		$\{\}$	$\star$		$\star$
$\mathbb{N}$	$+$	addition	0		$\star$		
$\mathbb{N}$	$\times$	multiplication	1	0	$\star$		

W a finite set, U an infinite set. For set Y, $\text{fin}(Y) \equiv \{X \in Y \mid X \text{ is finite}\}$

A few abstract semigroups

S	$\bullet$	description	α	ω	$\mathbb{CM}$	$\mathbb{SL}$	$\mathbb{IP}$
2^U	$\cup$	union	$\{\}$	U	$\star$		$\star$
2^U	$\cap$	intersection	U	$\{\}$	$\star$		$\star$
$2^{U\times U}$	$\bowtie$	relational join	$\mathcal{I}_U$	$\{\}$			
$X \rightarrow X$	$\circ$	composition	$\lambda x.x$				

U an infinite set

$X \bowtie Y \equiv \{(x, z) \in U \times U \mid \exists y \in U,\ (x,\ y) \in X \wedge (y,\ z) \in Y\}$

$\mathcal{I}_U \equiv \{(u,\ u) \mid u \in U\}$

subsemigroup

Suppose $(S,\ \bullet)$ is a semigroup and $T \subseteq S$. If T is closed w.r.t $\bullet$ (that is, $\forall x, y \in T, x \bullet y \in T$), then $(T,\ \bullet)$ is a subsemigroup of S.

Isomorphism

Reminder of function terminology

- $f \in X \to Y$
- f is injective (one-to-one) $\equiv \forall x, y \in X,\ f(x) = f(y) \implies x = y$
- f is surjective (onto) $\equiv \forall y \in Y,\ \exists x \in X,\ f(x) = y$
- f is bijective $\equiv$ f is injective and f is surjective

Isomorphism

If S and T are algebraic structures, then they are said to be isomorphic, written $S \approx T$, if there exists a bijective funtion $f \in S \to T$ which preserves structure.

Semigroup Isomorphism $S \approx T$

- $(S, \bullet)$ a semigroup
- $(T, \diamond)$ a semigroup
- $f \in S \to T$ a bijection
- $\forall a, b \in S,\ f(a \bullet b) = f(a) \diamond f(b)$

Cayley's Theorem for Semigroups

Every semigroup $(S, \bullet)$ is isomorphic to a subsemigroup of $(S \to S, \circ)$.

Partial proof of Cayley's theorem

$$\begin{array}{rcl} f_s(a) & \equiv & s \bullet a \\ T & \equiv & \{f_s \mid s \in S\} \subseteq S \to S \end{array} \qquad \begin{array}{rcl} \phi & \in & S \to T \\ \phi(s) & \equiv & f_s \end{array}$$

$\phi(s \bullet t) = \phi(s) \circ \phi(t)$

$$\begin{array}{rcl} f_{s \bullet t}(a) & = & (s \bullet t) \bullet a \\ & = & s \bullet (t \bullet a) \\ & = & s \bullet f_t(a) \\ & = & f_s(f_t(a)) \\ & = & (f_s \circ f_t)(a) \end{array}$$

Wait, is it injective?

$$f_s = f_t \Leftrightarrow \forall a \in S,\ f_s(a) = f_t(a) \Leftrightarrow \forall a \in S,\ s \bullet a = t \bullet a$$

But we want $s = t$! If there is an identity $\alpha \in S$, then letting $a = \alpha$ we have $s \bullet \alpha = t \bullet \alpha$, that is $s = t$.

But when there is no identity? (See Homework 1.)

Add identity

$$\text{AddId}(\alpha,\ (S,\ \bullet)) \equiv (S \uplus \{\alpha\}, \bullet_{\alpha}^{\text{id}})$$

where

$$a \bullet_{\alpha}^{\text{id}} b \quad \equiv \quad \begin{cases} a & (\text{if } b = \text{inr}(\alpha)) \\ b & (\text{if } a = \text{inr}(\alpha)) \\ \text{inl}(x \bullet y) & (\text{if } a = \text{inl}(x),\ b = \text{inl}(y)) \end{cases}$$

disjoint union

$$A \uplus B \equiv \{\text{inl}(a) \mid a \in A\} \cup \{\text{inr}(b) \mid b \in B\}$$

Add identity

Easy Exercises

$$
\begin{array}{rcl}
\mathbb{AS}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\
\mathbb{ID}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{AN}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \\
\mathbb{CM}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\
\mathbb{IP}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\
\mathbb{SL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet)
\end{array}
$$

Inserting an annihilator

$$\mathrm{AddAn}(\omega,\ (S,\ \bullet)) \equiv (S \uplus \{\omega\}, \bullet_{\omega}^{\mathrm{an}})$$

where

$$a \bullet_{\omega}^{\mathrm{an}} b \quad \equiv \quad \begin{cases} \mathrm{inr}(\omega) & (\text{if } b = \mathrm{inr}(\omega)) \\ \mathrm{inr}(\omega) & (\text{if } a = \mathrm{inr}(\omega)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

Add annihilator

Easy Exercises

$$
\begin{array}{rcl}
\mathbb{AS}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\
\mathbb{ID}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \\
\mathbb{AN}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{CM}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\
\mathbb{IP}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\
\mathbb{SL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet)
\end{array}
$$

Lexicographic Product of Semigroups

Lexicographic product semigroup

Suppose that semigroup $(S, \bullet)$ is commutative, idempotent, and selective and that $(T, \diamond)$ is a semigroup.

$$(S, \bullet) \; \vec{\times} \; (T, \diamond) \equiv (S \times T, \star)$$

where $\star \equiv \bullet \vec{\times} \diamond$ is defined as

$$(s_1, t_1) \star (s_2, t_2) = \begin{cases} (s_1 \bullet s_2, \; t_1 \diamond t_2) & s_1 = s_1 \bullet s_2 = s_2 \\ (s_1 \bullet s_2, \; t_1) & s_1 = s_1 \bullet s_2 \neq s_2 \\ (s_1 \bullet s_2, \; t_2) & s_1 \neq s_1 \bullet s_2 = s_2 \end{cases}$$

Examples

$(\mathbb{N}, \min) \ \vec{\times} \ (\mathbb{N}, \min)$

$$
\begin{array}{rcl}
(1,\ 17) \star (2,3) & = & (1,17) \\
(2,\ 17) \star (2,3) & = & (2,3) \\
(2,\ 3) \star (2,3) & = & (2,3)
\end{array}
$$

$(\mathbb{N}, \min) \ \vec{\times} \ (\mathbb{N}, \max)$

$$
\begin{array}{rcl}
(1,\ 17) \star (2,3) & = & (1,17) \\
(2,\ 17) \star (2,3) & = & (2,17) \\
(2,\ 3) \star (2,3) & = & (2,3)
\end{array}
$$

$(\mathbb{N}, \max) \ \vec{\times} \ (\mathbb{N}, \min)$

$$
\begin{array}{rcl}
(1,\ 17) \star (2,3) & = & (2,3) \\
(2,\ 17) \star (2,3) & = & (2,3) \\
(2,\ 3) \star (2,3) & = & (2,3)
\end{array}
$$

Assuming $\mathbb{AS}(S, \bullet) \wedge \mathbb{CM}(S, \bullet) \wedge \mathbb{IP}(S, \bullet) \wedge \mathbb{SL}(S, \bullet)$

$$
\begin{array}{rcl}
\mathbb{AS}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{AS}(T, \diamond) \\
\mathbb{ID}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \wedge \mathbb{ID}(T, \diamond) \\
\mathbb{AN}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \wedge \mathbb{AN}(T, \diamond) \\
\mathbb{CM}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{CM}(T, \diamond) \\
\mathbb{IP}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{IP}(T, \diamond) \\
\mathbb{SL}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{SL}(T, \diamond) \\
\mathbb{IR}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

All easy, except for $\mathbb{AS}$ (See Homework 1!).

Direct Product of Semigroups

Let $(S, \bullet)$ and $(T, \diamond)$ be semigroups.

Definition (Direct product semigroup)

The direct product is denoted

$$(S, \bullet) \times (T, \diamond) \equiv (S \times T, \star)$$

where

$$\star = \bullet \times \diamond$$

is defined as

$$(s_1, t_1) \star (s_2, t_2) = (s_1 \bullet s_2, t_1 \diamond t_2).$$

Easy exercises

$$
\begin{array}{rcl}
\mathbb{AS}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \wedge \mathbb{AS}(T, \diamond) \\
\mathbb{ID}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \wedge \mathbb{ID}(T, \diamond) \\
\mathbb{AN}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \wedge \mathbb{AN}(T, \diamond) \\
\mathbb{CM}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \wedge \mathbb{CM}(T, \diamond) \\
\mathbb{IP}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \wedge \mathbb{IP}(T, \diamond)
\end{array}
$$

What about $\mathbb{SL}$?

Consider the product of two selective semigroups, such as $(\mathbb{N}, \ \min) \times (\mathbb{N}, \ \max)$.

$$(10, \ 10) \star (1, \ 3) = (1, \ 10) \notin \{(10, \ 10), \ (1, \ 3)\}$$

The result in this case is not selective!

Direct product and $\mathbb{SL}$?

$$\mathbb{SL}((S, \bullet) \times (T, \diamond)) \quad \Leftrightarrow \quad (\mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond)) \vee (\mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond))$$

$$\begin{array}{rrcl} \mathbb{IR} & \text{is right} & \equiv & \forall s, t \in S, s \bullet t = t \\ \mathbb{IL} & \text{is left} & \equiv & \forall s, t \in S, s \bullet t = s \end{array}$$

See Homework 1

$$\begin{array}{lcl} \mathbb{IR}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond) \\ \mathbb{IL}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond) \end{array}$$

Revisit other constructions ...

$$
\begin{array}{rcl}
\mathbb{IR}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

$$
\begin{array}{rcl}
\mathbb{IR}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

Assuming $\mathbb{AS}(S, \bullet) \wedge \mathbb{CM}(S, \bullet) \wedge \mathbb{IP}(S, \bullet) \wedge \mathbb{SL}(S, \bullet)$

$$
\begin{array}{rcl}
\mathbb{IR}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

Lifted Product

Lifted product semigroup

Assume $(S, \bullet)$ is a semigroup. Let $\mathrm{lift}(S, \bullet) \equiv (\mathrm{fin}(2^S), \hat{\bullet})$ where

$$X \hat{\bullet} Y = \{x \bullet y \mid x \in X, \ y \in Y\}.$$

$$\{1, 3, 17\} \hat{+} \{1, 3, 17\} = \{2, 4, 6, 18, 20, 34\}$$

$$
\begin{array}{rcl}
\mathbb{AS}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\
\mathbb{ID}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \;\; (\hat{\alpha} = \{\alpha\}) \\
\mathbb{AN}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{TRUE} \;\; (\omega = \{\}) \\
\mathbb{CM}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\
\mathbb{SL}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \;\wedge \mid S \mid = 2) \\
\mathbb{IP}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{SL}((S, \bullet)) \\
\mathbb{IL}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IR}(\mathrm{lift}(S, \bullet)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}
$$

Why bother with all of these $\Leftrightarrow$ rules?

I would rather calculate than prove!

$$
\begin{array}{ll}
 & \mathbb{P}(\mathrm{lift}(\mathrm{lift}(\{t,\ f\},\ \wedge)) \\
\Leftrightarrow & \mathbb{SL}(\{t,\ f\},\ \wedge) \\
\Leftrightarrow & \mathbb{IL}(\{t,\ f\},\ \wedge) \vee \mathbb{IR}(\{t,\ f\},\ \wedge) \vee (\mathbb{IP}(\{t,\ f\},\ \wedge)\ \wedge \mid \{t,\ f\} \mid = 2) \\
\Leftrightarrow & \mathbb{FALSE} \vee \mathbb{FALSE} \vee (\mathbb{TRUE} \wedge \mathbb{TRUE}) \\
\Leftrightarrow & \mathbb{TRUE}
\end{array}
$$

Note

This kind of calculation will become more interesting as we introduce more complex constructors and consider more complex properties — such as those associated with semirings.

Homework 1

Each question is 25 points.

1. Finish the proof of Cayley's theorem.
2. Prove
$$\mathbb{SL}((S, \bullet) \times (T, \diamond))$$
$$\Leftrightarrow$$
$$(\mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond)) \vee (\mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond))$$
3. Assume that $\mathbb{AS}(S, \bullet)$, $\mathbb{AS}(T, \diamond)$, $\mathbb{CM}(S, \bullet)$, $\mathbb{IP}(S, \bullet)$, and $\mathbb{SL}(S, \bullet)$ hold. Prove that $\mathbb{AS}((S, \bullet) \vec{\times} (T, \diamond))$. Did you really need $\mathbb{CM}(S, \bullet)$?
4. (Rather difficult). Prove
$$\mathbb{SL}(\mathrm{lift}(S, \bullet))$$
$$\Leftrightarrow$$
$$\mathbb{IL}(S, \bullet) \vee \mathbb{IR}(S, \bullet) \vee (\mathbb{IP}(S, \bullet) \ \wedge \mid S \mid = 2)$$

Bi-semigroups and Pre-Semirings

$(S, \oplus, \otimes)$ is a bi-semigroup when

- $(S, \oplus)$ is a semigroup
- $(S, \otimes)$ is a semigroup

$(S, \oplus, \otimes)$ is a pre-semiring when

- $(S, \oplus, \otimes)$ is a bi-semigroup
- $\oplus$ is commutative

and left- and right-distributivity hold,

$$\begin{array}{lcrcl} \mathbb{LD} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\ \mathbb{RD} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c) \end{array}$$

Semirings

$(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring when

- $(S, \oplus, \otimes)$ is a pre-semiring
- $(S, \oplus, \overline{0})$ is a (commutative) monoid
- $(S, \otimes, \overline{1})$ is a monoid
- $\overline{0}$ is an annihilator for $\otimes$

Examples

Pre-semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
min_plus	$\mathbb{N}$	min	$+$		0
max_min	$\mathbb{N}$	max	min	0	

Semirings

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0
bw	$\mathbb{N}^{\infty}$	max	min	0	∞

Note the sloppiness — the symbols $+$, max, and min in the two tables represent different functions....

How about $(\max, +)$?

Pre-semiring

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
max_plus	$\mathbb{N}$	max	$+$	0	0

- What about "$\overline{0}$ is an annihilator for $\otimes$"? No!

Fix that ...

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$
$\text{max\_plus}^{-\infty}$	$\mathbb{N} \uplus \{-\infty\}$	max	$+$	$-\infty$	0

Matrix Semirings

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- Define the semiring of $n \times n$-matrices over S : $(\mathbb{M}_n(S), \oplus, \otimes, \mathbf{J}, \mathbf{I})$

$\oplus$ and $\otimes$

$$
\begin{aligned}
(\mathbf{A} \oplus \mathbf{B})(i, j) &= \mathbf{A}(i, j) \oplus \mathbf{B}(i, j) \\
(\mathbf{A} \otimes \mathbf{B})(i, j) &= \bigoplus_{1 \le q \le n} \mathbf{A}(i, q) \otimes \mathbf{B}(q, j)
\end{aligned}
$$

J and I

$$
\begin{aligned}
\mathbf{J}(i, j) &= \overline{0} \\
\mathbf{I}(i, j) &= \begin{cases} \overline{1} & (\text{if } i = j) \\ \overline{0} & (\text{otherwise}) \end{cases}
\end{aligned}
$$

$\mathbb{M}_n(S)$ is a semiring!

For example, here is left distribution

$$\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C})$$

$$\begin{aligned}
& (\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}))(i,\ j) \\
= & \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes (\mathbf{B} \oplus \mathbf{C})(q,\ j) \\
= & \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes (\mathbf{B}(q,\ j) \oplus \mathbf{C}(q,\ j)) \\
= & \bigoplus_{1 \le q \le n} (\mathbf{A}(i,\ q) \otimes \mathbf{B}(q,\ j)) \oplus (\mathbf{A}(i,\ q) \otimes \mathbf{C}(q,\ j)) \\
= & (\bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{B}(q,\ j)) \oplus (\bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{C}(q,\ j)) \\
= & ((\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C}))(i,\ j)
\end{aligned}$$

Note : we only needed left-distributivity on S.

Matrix encoding path problems

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- $G = (V, E)$ a directed graph
- $w \in E \rightarrow S$ a weight function

Path weight

The <u>weight</u> of a path $p = i_1, i_2, i_3, \cdots, i_k$ is

$$w(p) = w(i_1, i_2) \otimes w(i_2, i_3) \otimes \cdots \otimes w(i_{k-1}, i_k).$$

The empty path is given the weight $\overline{1}$.

Adjacency matrix **A**

$$\mathbf{A}(i, j) = \begin{cases} w(i, j) & \text{if } (i, j) \in E, \\ \overline{0} & \text{otherwise} \end{cases}$$

The general problem of finding globally optimal path weights

Given an adjacency matrix $\mathbf{A}$, find $\mathbf{A}^*$ such that for all $i,\ j \in V$

$$\mathbf{A}^*(i,\ j) = \bigoplus_{p \in P(i,\ j)} w(p)$$

where $P(i,\ j)$ represents the set of all paths from i to j.

How can we solve this problem?

Matrix methods

Matrix powers, $\mathbf{A}^k$

$$\mathbf{A}^0 = \mathbf{I}$$

$$\mathbf{A}^{k+1} = \mathbf{A} \otimes \mathbf{A}^k$$

Closure, $\mathbf{A}^*$

$$\mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

$$\mathbf{A}^* = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \oplus \cdots$$

Note: $\mathbf{A}^*$ might not exist. Why?

Matrix methods can compute optimal path weights

- Let $P(i, j)$ be the set of paths from i to j.
- Let $P^k(i, j)$ be the set of paths from i to j with exactly k arcs.
- Let $P^{(k)}(i, j)$ be the set of paths from i to j with at most k arcs.

Theorem

$$
\begin{array}{lrcl}
(1) & \mathbf{A}^k(i,\ j) & = & \bigoplus_{p \in P^k(i,\ j)} w(p) \\
(2) & \mathbf{A}^{(k)}(i,\ j) & = & \bigoplus_{p \in P^{(k)}(i,\ j)} w(p) \\
(3) & \mathbf{A}^*(i,\ j) & = & \bigoplus_{p \in P(i,\ j)} w(p)
\end{array}
$$

Warning again: for some semirings the expression $\mathbf{A}^*(i,\ j)$ might not be well-defeind. Why?

Proof of (1)

By induction on k. Base Case: $k = 0$.

$$P^0(i,\ i) = \{\epsilon\},$$

so $\mathbf{A}^0(i\ ,i) = \mathbf{I}(i,\ i) = \overline{1} = w(\epsilon)$.

And $i \neq j$ implies $P^0(i,j) = \{\}$. By convention

$$\bigoplus_{p \in \{\}} w(p) = \overline{0} = \mathbf{I}(i,\ j).$$

Proof of (1)

Induction step.

$$
\begin{aligned}
\mathbf{A}^{k+1}(i,j) &= (\mathbf{A} \otimes \mathbf{A}^{k})(i,\ j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{A}^{k}(q,\ j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \Big(\bigoplus_{p \in P^{k}(q,\ j)} w(p) \Big) \\
&= \bigoplus_{1 \le q \le n} \bigoplus_{p \in P^{k}(q,\ j)} \mathbf{A}(i,\ q) \otimes w(p) \\
&= \bigoplus_{(i,\ q) \in E} \bigoplus_{p \in P^{k}(q,j)} w(i,\ q) \otimes w(p) \\
&= \bigoplus_{p \in P^{k+1}(i,\ j)} w(p)
\end{aligned}
$$

When does $\mathbf{A}^*$ exist? Try a general approach.

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring

Powers, a^k

$$\begin{array}{rcl} a^0 & = & \overline{1} \\ a^{k+1} & = & a \otimes a^k \end{array}$$

Closure, a^*

$$\begin{array}{rcl} a^{(k)} & = & a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \\ a^* & = & a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \oplus \cdots \end{array}$$

Definition (q stability)

If there exists a q such that $a^{(q)} = a^{(q+1)}$, then a is q-stable. By induction: $\forall t, 0 \leq t, a^{(q+t)} = a^{(q)}$. Therefore, $a^* = a^{(q)}$.

Fun Facts

Fact 1

If $\overline{1}$ is an annihiltor for $\oplus$, then every $a \in S$ is 0-stable!

Fact 2

If S is 0-stable, then $\mathbb{M}_n(S)$ is $(n-1)$-stable. That is,

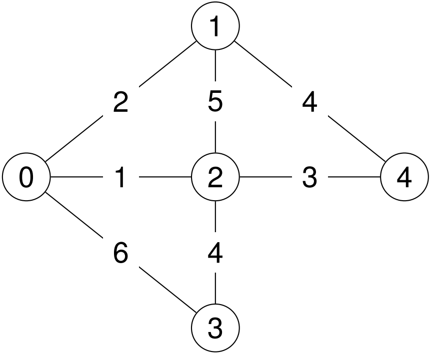
$$\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{n-1}$$

Why? Because we can ignore paths with loops.

$$(a \otimes c \otimes b) \oplus (a \otimes b) = a \otimes (\overline{1} \oplus c) \otimes b = a \otimes \overline{1} \otimes b = a \otimes b$$

Think of c as the weight of a loop in a path with weight $a \otimes b$.

Shortest paths example, $(\mathbb{N}^{\infty}, \min, +)$



The adjacency matrix

$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{bmatrix} \end{array}$$

Note that the longest shortest path is $(1, 0, 2, 3)$ of length 3 and weight 7.

$(\min, +)$ example

Our theorem tells us that $\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{A}^{(4)}$

$$\mathbf{A}^* = \mathbf{A}^{(4)} = \mathbf{I} \ \min \ \mathbf{A} \ \min \ \mathbf{A}^2 \ \min \ \mathbf{A}^3 \ \min \ \mathbf{A}^4 = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array} \right] \end{array}$$

$(\min, +)$ example

$$\mathbf{A} = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \infty & \underline{2} & \underline{1} & 6 & \infty \\ 1 & \underline{2} & \infty & 5 & \infty & \underline{4} \\ 2 & \underline{1} & 5 & \infty & \underline{4} & \underline{3} \\ 3 & 6 & \infty & \underline{4} & \infty & \infty \\ 4 & \infty & \underline{4} & \underline{3} & \infty & \infty \end{array} \qquad \mathbf{A}^3 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 8 & 4 & 3 & 8 & 10 \\ 1 & 4 & 8 & 7 & \underline{7} & 6 \\ 2 & 3 & 7 & 8 & 6 & 5 \\ 3 & 8 & \underline{7} & 6 & 11 & 10 \\ 4 & 10 & 6 & 5 & 10 & 12 \end{array}$$

$$\mathbf{A}^2 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 2 & 6 & 7 & \underline{5} & \underline{4} \\ 1 & 6 & 4 & \underline{3} & 8 & 8 \\ 2 & 7 & \underline{3} & 2 & 7 & 9 \\ 3 & \underline{5} & 8 & 7 & 8 & \underline{7} \\ 4 & \underline{4} & 8 & 9 & \underline{7} & 6 \end{array} \qquad \mathbf{A}^4 = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 4 & 8 & 9 & 7 & 6 \\ 1 & 8 & 6 & 5 & 10 & 10 \\ 2 & 9 & 5 & 4 & 9 & 11 \\ 3 & 7 & 10 & 9 & 10 & 9 \\ 4 & 6 & 10 & 11 & 9 & 8 \end{array}$$

First appearance of final value is in red and underlined. Remember: we are looking at all paths of a given length, even those with cycles!

A “better” way — our basic algorithm

$$\begin{aligned} \mathbf{A}^{\langle 0 \rangle} &= \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

Lemma

$$\mathbf{A}^{\langle k \rangle} = \mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^{1} \oplus \mathbf{A}^{2} \oplus \cdots \oplus \mathbf{A}^{k}$$

back to $(\min, +)$ example

$$\mathbf{A}^{\langle 1 \rangle} = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} 0 & 2 & 1 & 6 & \infty \\ 2 & 0 & 5 & \infty & 4 \\ 1 & 5 & 0 & 4 & 3 \\ 6 & \infty & 4 & 0 & \infty \\ \infty & 4 & 3 & \infty & 0 \end{array}\right] \end{array}$$

$$\mathbf{A}^{\langle 3 \rangle} = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array}\right] \end{array}$$

$$\mathbf{A}^{\langle 2 \rangle} = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \left[\begin{array}{ccccc} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 8 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 8 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{array}\right] \end{array}$$

A note on $\mathbf{A}$ vs. $\mathbf{A} \oplus \mathbf{I}$

Lemma

If $\oplus$ is idempotent, then

$$(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}.$$

Proof. Base case: When $k = 0$ both expressions are $\mathbf{I}$.
Assume $(\mathbf{A} \oplus \mathbf{I})^k = \mathbf{A}^{(k)}$. Then

$$\begin{array}{rcl}
(\mathbf{A} \oplus \mathbf{I})^{k+1} & = & (\mathbf{A} \oplus \mathbf{I})(\mathbf{A} \oplus \mathbf{I})^k \\
& = & (\mathbf{A} \oplus \mathbf{I})\mathbf{A}^{(k)} \\
& = & \mathbf{A}\mathbf{A}^{(k)} \oplus \mathbf{A}^{(k)} \\
& = & \mathbf{A}(\mathbf{I} \oplus \mathbf{A} \oplus \cdots \oplus \mathbf{A}^k) \oplus \mathbf{A}^{(k)} \\
& = & \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
& = & \mathbf{A}^{k+1} \oplus \mathbf{A}^{(k)} \\
& = & \mathbf{A}^{(k+1)}
\end{array}$$

Order Relations

We are interested in order relations $\leq \subseteq S \times S$

Definition (Important Order Properties)

$$
\begin{array}{llcl}
\mathbb{RX} & \text{reflexive} & \equiv & a \leq a \\
\mathbb{TR} & \text{transitive} & \equiv & a \leq b \wedge b \leq c \rightarrow a \leq c \\
\mathbb{AY} & \text{antisymmetric} & \equiv & a \leq b \wedge b \leq a \rightarrow a = b \\
\mathbb{TO} & \text{total} & \equiv & a \leq b \vee b \leq a
\end{array}
$$

	pre-order	partial order	preference order	total order
$\mathbb{RX}$	$\star$	$\star$	$\star$	$\star$
$\mathbb{TR}$	$\star$	$\star$	$\star$	$\star$
$\mathbb{AY}$		$\star$		$\star$
$\mathbb{TO}$			$\star$	$\star$

Canonical Pre-order of a Commutative Semigroup

Definition (Canonical pre-orders)

$$
\begin{aligned}
a \trianglelefteq_{\bullet}^{R} b &\equiv \exists c \in S : b = a \bullet c \\
a \trianglelefteq_{\bullet}^{L} b &\equiv \exists c \in S : a = b \bullet c
\end{aligned}
$$

Lemma (Sanity check)

Associativity of $\bullet$ implies that these relations are transitive.

Proof.

Note that $a \trianglelefteq_{\bullet}^{R} b$ means $\exists c_1 \in S : b = a \bullet c_1$, and $b \trianglelefteq_{\bullet}^{R} c$ means $\exists c_2 \in S : c = b \bullet c_2$. Letting $c_3 = c_1 \bullet c_2$ we have $c = b \bullet c_2 = (a \bullet c_1) \bullet c_2 = a \bullet (c_1 \bullet c_2) = a \bullet c_3$. That is, $\exists c_3 \in S : c = a \bullet c_3$, so $a \trianglelefteq_{\bullet}^{R} c$. The proof for $\trianglelefteq_{\bullet}^{L}$ is similar. $\square$

Canonically Ordered Semigroup

Definition (Canonically Ordered Semigroup)

A commutative semigroup $(S, \bullet)$ is canonically ordered when $a \trianglelefteq_{\bullet}^{R} c$ and $a \trianglelefteq_{\bullet}^{L} c$ are partial orders.

Definition (Groups)

A monoid is a group if for every $a \in S$ there exists a $a^{-1} \in S$ such that $a \bullet a^{-1} = a^{-1} \bullet a = \alpha$.

Canonically Ordered Semigroups vs. Groups

Lemma (THE BIG DIVIDE)

Only a trivial group is canonically ordered.

Proof.

If $a, \; b \in S$, then $a = \alpha_{\bullet} \bullet a = (b \bullet b^{-1}) \bullet a = b \bullet (b^{-1} \bullet a) = b \bullet c$, for $c = b^{-1} \bullet a$, so $a \trianglelefteq^{L}_{\bullet} b$. In a similar way, $b \trianglelefteq^{R}_{\bullet} a$. Therefore $a = b$. $\square$

Natural Orders

Definition (Natural orders)

Let $(S, \bullet)$ be a semigroup.

$$\begin{array}{lcl} a \leq^{L}_{\bullet} b & \equiv & a = a \bullet b \\ a \leq^{R}_{\bullet} b & \equiv & b = a \bullet b \end{array}$$

Lemma

If $\bullet$ is commutative and idempotent, then $a \trianglelefteq^{D}_{\bullet} b \iff a \leq^{D}_{\bullet} b$, for $D \in \{R, L\}$.

Proof.

$$\begin{array}{lcl} a \trianglelefteq^{R}_{\bullet} b & \iff & b = a \bullet c = (a \bullet a) \bullet c = a \bullet (a \bullet c) \\ & = & a \bullet b \iff a \leq^{R}_{\bullet} b \\ a \trianglelefteq^{L}_{\bullet} b & \iff & a = b \bullet c = (b \bullet b) \bullet c = b \bullet (b \bullet c) \\ & = & b \bullet a = a \bullet b \iff a \leq^{L}_{\bullet} b \end{array}$$

□

Special elements and natural orders

Lemma (Natural Bounds)

- *If α exists, then for all a, $a \leq_\bullet^L \alpha$ and $\alpha \leq_\bullet^R a$*
- *If ω exists, then for all a, $\omega \leq_\bullet^L a$ and $a \leq_\bullet^R \omega$*
- *If α and ω exist, then S is bounded.*

$$\begin{array}{ccccc} \omega & \leq_\bullet^L & a & \leq_\bullet^L & \alpha \\ \alpha & \leq_\bullet^R & a & \leq_\bullet^R & \omega \end{array}$$

Remark (Thanks to Iljitsch van Beijnum)

Note that this means for (min, $+$) we have

$$\begin{array}{ccccc} 0 & \leq_{\text{min}}^L & a & \leq_{\text{min}}^L & \infty \\ \infty & \leq_{\text{min}}^R & a & \leq_{\text{min}}^R & 0 \end{array}$$

and still say that this is bounded, even though one might argue with the terminology!

Examples of special elements

S	$\bullet$	α	ω	$\leq^{\mathrm{L}}_{\bullet}$	$\leq^{\mathrm{R}}_{\bullet}$
$\mathbb{N}^{\infty}$	min	∞	0	$\leq$	$\geq$
$\mathbb{N}^{-\infty}$	max	0	$-\infty$	$\geq$	$\leq$
$\mathcal{P}(W)$	$\cup$	$\{\}$	W	$\subseteq$	$\supseteq$
$\mathcal{P}(W)$	$\cap$	W	$\{\}$	$\supseteq$	$\subseteq$

Property Management

Lemma

Let $D \in \{R, L\}$.

1. $\mathbb{IP}(S, \bullet) \iff \mathbb{RX}(S, \leq_\bullet^D)$
2. $\mathbb{CM}(S, \bullet) \implies \mathbb{AY}(S, \leq_\bullet^D)$
3. $\mathbb{AS}(S, \bullet) \implies \mathbb{TR}(S, \leq_\bullet^D)$
4. $\mathbb{CM}(S, \bullet) \implies (\mathbb{SL}(S, \bullet) \iff \mathbb{TO}(S, \leq_\bullet^D))$

Proof.

1. $a \leq_\bullet^D a \iff a = a \bullet a$,
2. $a \leq_\bullet^L b \wedge b \leq_\bullet^L a \iff a = a \bullet b \wedge b = b \bullet a \implies a = b$
3. $a \leq_\bullet^L b \wedge b \leq_\bullet^L c \iff a = a \bullet b \wedge b = b \bullet c \implies a = a \bullet (b \bullet c) = (a \bullet b) \bullet c = a \bullet c \implies a \leq_\bullet^L c$
4. $a = a \bullet b \vee b = a \bullet b \iff a \leq_\bullet^L b \vee b \leq_\bullet^L a$

□

Bounds

Suppose $(S, \leq)$ is a partially ordered set.

greatest lower bound

For $a, b \in S$, the element $c \in S$ is the greatest lower bound of a and b, written $c = a \ \mathrm{glb} \ b$, if it is a lower bound ($c \leq a$ and $c \leq b$), and for every $d \in S$ with $d \leq a$ and $d \leq b$, we have $d \leq c$.

least upper bound

For $a, b \in S$, the element $c \in S$ is the least upper bound of a and b, written $c = a \ \mathrm{lub} \ b$, if it is an upper bound ($a \leq c$ and $b \leq c$), and for every $d \in S$ with $a \leq d$ and $b \leq d$, we have $c \leq d$.

Semi-lattices

Suppose $(S, \leq)$ is a partially ordered set.

meet-semilattice

S is a meet-semilattice if $a \text{ glb } b$ exists for each $a, b \in S$.

join-semilattice

S is a join-semilattice if $a \text{ lub } b$ exists for each $a, b \in S$.

Fun Facts

Fact 3

Suppose $(S, \bullet)$ is a commutative and idempotent semigroup.

- $(S, \leq^L_\bullet)$ is a meet-semilattice with $a \operatorname{glb} b = a \bullet b$.
- $(S, \leq^R_\bullet)$ is a join-semilattice with $a \operatorname{lub} b = a \bullet b$.

Fact 4

Suppose $(S, \leq)$ is a partially ordered set.

- If $(S, \leq)$ is a meet-semilattice, then (S, glb) is a commutative and idempotent semigroup.
- If $(S, \leq)$ is a join-semilattice, then (S, lub) is a commutative and idempotent semigroup.

That is, semi-lattices represent the same class of structures as commutative and idempotent semigroups.

Semigroup properties (so far)

$$
\begin{array}{rcl}
\mathbb{AS}(S,\ \bullet) & \equiv & \forall a, b, c \in S,\ a \bullet (b \bullet c) = (a \bullet b) \bullet c \\
\mathbb{IID}(S,\ \bullet,\ \alpha) & \equiv & \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha \\
\mathbb{ID}(S,\ \bullet) & \equiv & \exists \alpha \in S,\ \mathbb{IID}(S,\ \bullet,\ \alpha) \\
\mathbb{IAN}(S,\ \bullet,\ \omega) & \equiv & \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega \\
\mathbb{AN}(S,\ \bullet) & \equiv & \exists \omega \in S,\ \mathbb{IAN}(S,\ \bullet,\ \omega) \\
\mathbb{CM}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b = b \bullet a \\
\mathbb{SL}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b \in \{a,\ b\} \\
\mathbb{IP}(S,\ \bullet) & \equiv & \forall a \in S,\ a \bullet a = a \\
\mathbb{IR}(S,\ \bullet) & \equiv & \forall s, t \in S, s \bullet t = t \\
\mathbb{IL}(S,\ \bullet) & \equiv & \forall s, t \in S, s \bullet t = s
\end{array}
$$

Recall that is right ($\mathbb{IR}$) and is left ($\mathbb{IL}$) are forced on us by wanting an $\Leftrightarrow$-rule for $\mathbb{SL}((S, \bullet) \times (T, \diamond))$

Bisemigroup properties (so far)

$$
\begin{array}{rcl}
\mathbb{AAS}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{AS}(S,\ \oplus) \\
\mathbb{AID}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{ID}(S,\ \oplus) \\
\mathbb{ACM}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{CM}(S,\ \oplus) \\
\mathbb{MAS}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{AS}(S,\ \otimes) \\
\mathbb{MID}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{ID}(S,\ \otimes) \\
\mathbb{LD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \\
\mathbb{RD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \\
\mathbb{ZA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{0} \in S,\ \mathbb{IID}(S,\ \oplus,\ \overline{0}) \wedge \mathbb{IAN}(S,\ \otimes,\ \overline{0}) \\
\hline
\mathbb{OA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{1} \in S,\ \mathbb{IID}(S,\ \otimes,\ \overline{1}) \wedge \mathbb{IAN}(S,\ \oplus,\ \overline{1}) \\
\mathbb{ASL}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{SL}(S,\ \oplus) \\
\mathbb{AIP}(S,\ \oplus,\ \otimes) & \equiv & \mathbb{IP}(S,\ \oplus)
\end{array}
$$

Operations for adding a zero, a one

$$\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes)) \ \equiv\ (S \uplus \{\overline{0}\},\ \oplus^{\mathrm{id}}_{\overline{0}},\ \otimes^{\mathrm{an}}_{\overline{0}})$$

$$\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes)) \ \equiv\ (S \uplus \{\overline{1}\},\ \oplus^{\mathrm{an}}_{\overline{1}},\ \otimes^{\mathrm{id}}_{\overline{1}})$$

Recall

$$a \bullet^{\mathrm{id}}_{\alpha} b \ \equiv\ \begin{cases} a & (\text{if } b = \mathrm{inr}(\alpha)) \\ b & (\text{if } a = \mathrm{inr}(\alpha)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

$$a \bullet^{\mathrm{an}}_{\omega} b \ \equiv\ \begin{cases} \mathrm{inr}(\omega) & (\text{if } b = \mathrm{inr}(\omega)) \\ \mathrm{inr}(\omega) & (\text{if } a = \mathrm{inr}(\omega)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

We can “inherit” semigroup rules

Examples

$$\begin{array}{rcl} \mathbb{ACM}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \equiv & \mathbb{CM}(\mathrm{AddId}(\overline{0},\ (S,\ \oplus))) \\ & \Leftrightarrow & \mathbb{CM}(S,\ \oplus) \end{array}$$

$$\begin{array}{rcl} \mathbb{MID}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \equiv & \mathbb{ID}(\mathrm{AddAn}(\overline{0},\ (S,\ \otimes))) \\ & \Leftrightarrow & \mathbb{ID}(S,\ \otimes) \end{array}$$

Property management for AddZero

“Inherited” rules

$$
\begin{array}{rcl}
\mathbb{AAS}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{AS}(S,\ \oplus) \\
\mathbb{AID}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{ACM}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{CM}(S,\ \oplus) \\
\mathbb{ASL}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{SL}(S,\ \oplus) \\
\mathbb{AIP}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{IP}(S,\ \oplus) \\
\mathbb{MAS}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{AS}(S,\ \otimes) \\
\mathbb{MID}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ID}(S,\ \otimes)
\end{array}
$$

Easy Exercises

$$
\begin{array}{rcl}
\mathbb{LD}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \\
\mathbb{RD}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \\
\mathbb{ZA}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{OA}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{OA}(S,\ \oplus,\ \otimes)
\end{array}
$$

Easy Exercises?

Consider left distributivity ($\mathbb{LD}$)

a	b	c	$a \otimes_{\overline{0}}^{\text{an}} (b \oplus_{\overline{0}}^{\text{id}} c)$	$(a \otimes_{\overline{0}}^{\text{an}} b) \oplus_{\overline{0}}^{\text{id}} (a \otimes_{\overline{0}}^{\text{an}} c)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inl}(a' \otimes (b' \oplus c'))$	$\text{inl}((a' \otimes b') \oplus (a' \otimes c'))$
$\text{inr}(\overline{0})$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$
$\text{inl}(a')$	$\text{inr}(\overline{0})$	$\text{inl}(c')$	$\text{inl}(a' \oplus c')$	$\text{inl}(a' \oplus c')$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inr}(\overline{0})$	$\text{inl}(a' \oplus b')$	$\text{inl}(a' \oplus b')$
$\text{inl}(a')$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$
$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$

However, adding a one is more complicated!

Consider left distributivity ($\mathbb{LD}$)

a	b	c	$a \otimes^{\mathrm{id}}_{1} (b \oplus^{\mathrm{an}}_{1} c)$	$(a \otimes^{\mathrm{id}}_{1} b) \oplus^{\mathrm{an}}_{1} (a \otimes^{\mathrm{id}}_{1} c)$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inl}(c')$	$\mathrm{inl}(a' \otimes (b' \oplus c'))$	$\mathrm{inl}((a' \otimes b') \oplus (a' \otimes c'))$
$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(b')$	$\mathrm{inl}(c')$	$\mathrm{inl}(b' \oplus c')$	$\mathrm{inl}(b' \oplus c')$
$\mathrm{inl}(a')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(c')$	$\mathrm{inl}(a')$	$\mathrm{inl}((a' \oplus (a' \otimes c'))$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(a')$	$\mathrm{inl}((a' \otimes b') \oplus a')$
$\mathrm{inl}(a')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(a')$	$\mathrm{inl}(a' \oplus a')$
$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$

What is this?

$a = (a \otimes b) \oplus a$

Suppose $\oplus$ is idempotent and commutative and we let $a \leq b \equiv a = a \oplus b$. We know that

$$b \leq c \Rightarrow a \otimes b \leq a \otimes c$$

since $b = b \oplus c$ implies $a \otimes b = a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$. That is $\otimes$ is order preserving.
Now $a = (a \otimes b) \oplus a$ is telling us something else, that

$$a \leq a \otimes b.$$

That is, that multiplication is <u>inflationary</u>.

Absorption

$\mathbb{AB}$sorption properties (name is from lattice theory)

$$\begin{array}{rcl} \mathbb{RAB}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b \in S,\ a = (a \otimes b) \oplus a = a \oplus (a \otimes b) \\ \mathbb{LAB}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b \in S,\ a = (b \otimes a) \oplus a = a \oplus (b \otimes a) \end{array}$$

Observations

$$\begin{array}{rcl} \mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{ID}(S,\ \oplus) & \Rightarrow & \mathbb{IP}(S,\ \otimes) \\ \mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{ID}(S,\ \oplus) & \Rightarrow & \mathbb{IP}(S,\ \otimes) \\ \mathbb{LD}(S,\ \oplus,\ \otimes) \wedge \mathbb{OA}(S,\ \oplus,\ \otimes) & \Rightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \\ \mathbb{RD}(S,\ \oplus,\ \otimes) \wedge \mathbb{OA}(S,\ \oplus,\ \otimes) & \Rightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes) \end{array}$$

Rules for absorption? Consider $\mathbb{RAB}$

AddZero

a	b	$(a \otimes_{0}^{\text{an}} b) \oplus_{0}^{\text{id}} a$	$a \oplus_{0}^{\text{id}} (a \otimes_{0}^{\text{an}} b)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}((a' \otimes b') \oplus a)$	$\text{inl}(a' \oplus (a' \otimes b'))$
$\text{inr}(\overline{0})$	$\text{inl}(b')$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$
$\text{inl}(a')$	$\text{inr}(\overline{0})$	$\text{inl}(a')$	$\text{inl}(a')$
$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$

$$\mathbb{RAB}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) \ \Leftrightarrow\ \mathbb{RAB}(S,\ \oplus,\ \otimes)$$
$$\mathbb{LAB}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) \ \Leftrightarrow\ \mathbb{LAB}(S,\ \oplus,\ \otimes)$$

Rules for absorption? Consider $\mathbb{RAB}$

AddOne

a	b	$(a \otimes_{\overline{1}}^{\mathrm{id}} b) \oplus_{\overline{1}}^{\mathrm{an}} a$	$a \oplus_{\overline{1}}^{\mathrm{an}} (a \otimes_{\overline{1}}^{\mathrm{id}} b)$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inl}((a' \otimes b') \oplus a)$	$\mathrm{inl}(a' \oplus (a' \otimes b'))$
$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(b')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$
$\mathrm{inl}(a')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(a')$	$\mathrm{inl}(a' \oplus a')$
$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$

Property management for AddOne

"Inherited" rules

$$
\begin{array}{rcl}
\mathbb{AAS}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{AS}(S,\ \oplus) \\
\mathbb{AID}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ID}(S,\ \oplus) \\
\mathbb{ACM}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{CM}(S,\ \oplus) \\
\mathbb{ASL}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{SL}(S,\ \oplus) \\
\mathbb{AIP}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{IP}(S,\ \oplus) \\
\mathbb{MAS}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{AS}(S,\ \otimes) \\
\mathbb{MID}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE}
\end{array}
$$

Property management for AddOne

$$
\begin{array}{rcl}
\mathbb{LD}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \wedge \mathbb{RAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{RD}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \wedge \mathbb{LAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{ZA}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ZA}(S,\ \oplus,\ \otimes) \\
\mathbb{OA}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{RAB}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus) \\
\mathbb{LAB}(\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus)
\end{array}
$$

We have to start somewhere!

S	$\oplus$	$\otimes$	$\overline{0}$	$\overline{1}$	$\mathbb{LD}$	$\mathbb{RD}$	$\mathbb{ZA}$	$\mathbb{OA}$	$\mathbb{LAB}$	$\mathbb{RAB}$
$\mathbb{N}$	min	$+$		0	$\star$	$\star$		$\star$	$\star$	$\star$
$\mathbb{N}$	max	$+$	0	0	$\star$	$\star$			$\star$	$\star$
$\mathbb{N}$	max	min	0		$\star$	$\star$	$\star$		$\star$	$\star$
$\mathbb{N}$	min	max		0	$\star$	$\star$		$\star$	$\star$	$\star$

Introducing Minimax

$$\begin{aligned}\text{minimax} &\equiv \text{AddZero}(\infty,\ (\mathbb{N},\ \mathsf{min},\ \mathsf{max})) \\ &= (\mathbb{N} \uplus \{\infty\},\ \mathsf{min}^{\text{id}}_{\infty},\ \mathsf{max}^{\text{an}}_{\infty})\end{aligned}$$

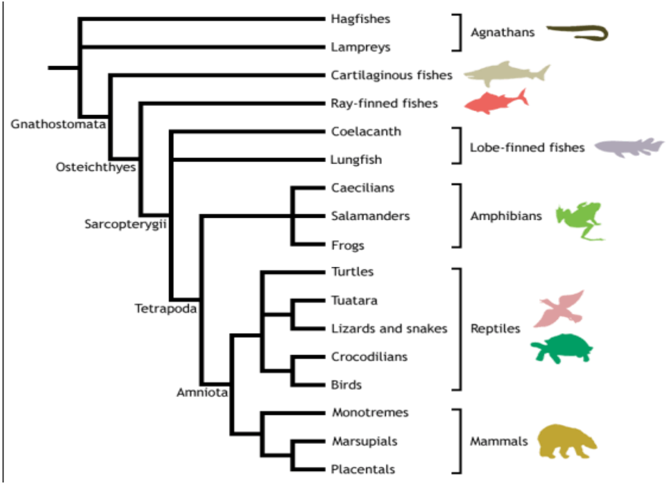
Some examples ...

$$\begin{aligned}\text{inl}(17)\ \mathsf{min}^{\text{id}}_{\infty}\ \text{inr}(\infty) &= \text{inl}(17) \\ \text{inl}(17)\ \mathsf{max}^{\text{an}}_{\infty}\ \text{inr}(\infty) &= \text{inr}(\infty)\end{aligned}$$

... which we will usually write as

$$\begin{aligned}17\,\mathsf{min}\,\infty &= 17 \\ 17\,\mathsf{max}\,\infty &= \infty\end{aligned}$$

Dendrograms



http://www.instituteofcaninebiology.org/how-to-read-a-dendrogram.html

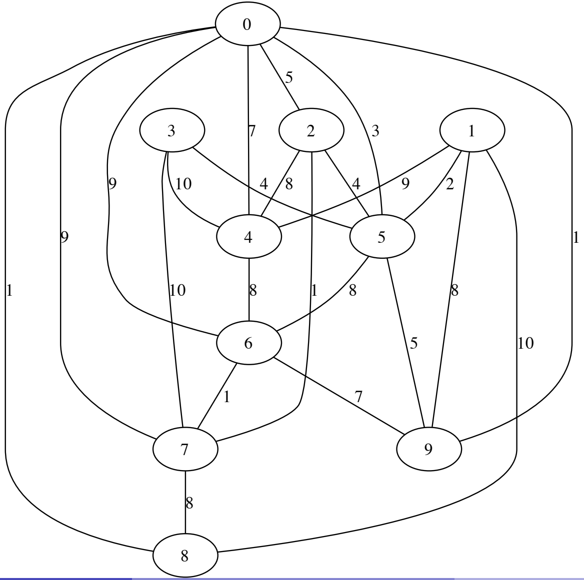
An application of Minimax

- Given an adjacency matrix $\mathbf{A}$ over $\mathrm{minimax}$,
- suppose that $\mathbf{A}(i,\ j) = 0 \Leftrightarrow i = j$,
- suppose that $\mathbf{A}$ is symmetric ($\mathbf{A}(i,\ j) = \mathbf{A}(j,\ i)$),
- interpret $\mathbf{A}(i,\ j)$ as <u>measured</u> dissimilarity of i and j,
- interpret $\mathbf{A}^*(i,\ j)$ as <u>inferred</u> dissimilarity of i and j,

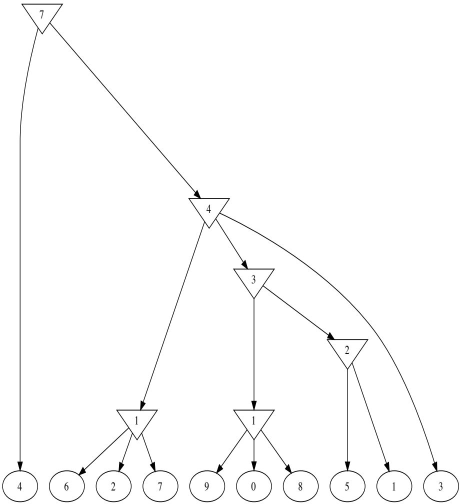
Many uses

- Hierarchical clustering of large data sets
- Classification in Machine Learning
- Computational phylogenetic
- ...

A (random) minimax matrix **A** drawn as a graph



The solution $\mathbf{A}^*$ drawn as a dendrogram



Hierarchical clustering? Why?

Suppose $(Y, \leq, +)$ is a totally ordered with least element 0.

Metric

A metric for set X over $(Y, \leq, +)$ is a function $d \in X \times X \rightarrow Y$ such that

- $\forall x, y \in X,\ d(x,\ y) = 0 \Leftrightarrow x = y$
- $\forall x, y \in X,\ d(x,\ y) = d(y,\ x)$
- $\forall x, y, z \in X,\ d(x,\ y) \leq d(x,\ z) + d(z,\ y)$

Ultrametric

An ultrametric for set X over $(Y, \leq)$ is a function $d \in X \times X \rightarrow Y$ such that

- $\forall x \in X,\ d(x,\ x) = 0$
- $\forall x, y \in X,\ d(x,\ y) = d(y,\ x)$
- $\forall x, y, z \in X,\ d(x,\ y) \leq d(x,\ z) \max d(z,\ y)$

Fun Facts

$\mathrm{minimax}$ and ultrametrics

If $\mathbf{A}$ is an $n \times n$ symmetric $\mathrm{minimax}$ adjacency matrix, then $\mathbf{A}^*$ is a finite ultrametric for $\{0,\ 1,\ \ldots,\ n-1\}$ over $(\mathbb{N}^\infty,\ \leq))$.

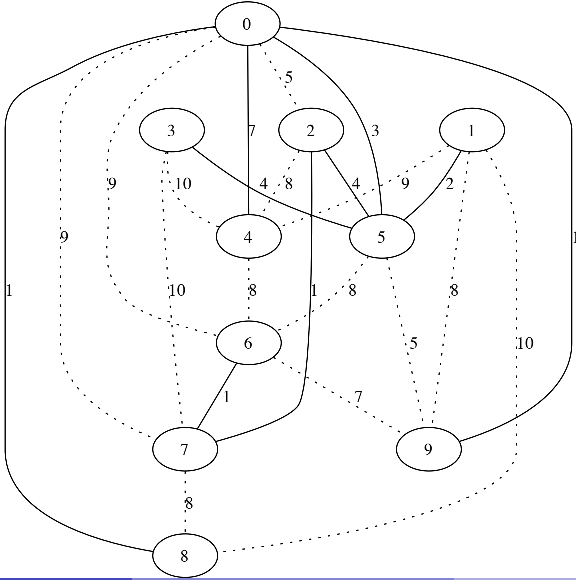
$\mathrm{minimax}$ and spanning trees

The set of arcs

$$\{(i,\ j) \in E \mid \mathbf{A}(i,\ j) = \mathbf{A}^*(i,\ j)\}$$

contain a spanning tree

A spanning tree derived from **A** and **A***



Recall

Lexicographic Product of Semigroups

Suppose that

$$\mathbb{AS}(S,\ \oplus_S) \wedge \mathbb{CM}(S,\ \oplus_S) \wedge \mathbb{SL}(S,\ \oplus_S) \wedge \mathbb{AS}(T,\ \oplus_T).$$

Let

$$(S,\ \oplus_S) \vec{\times} (T,\ \oplus_T) \equiv (S \times T,\ \oplus_S \vec{\times} \oplus_T)$$

where

$$(s_1,\ t_1)\ \oplus_S \vec{\times} \oplus_T\ (s_2,\ t_2) \equiv \begin{cases} (s_1 \oplus_S s_2,\ t_1 \oplus_T t_2) & s_1 = s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2,\ t_1) & s_1 = s_1 \oplus_S s_2 \neq s_2 \\ (s_1 \oplus_S s_2,\ t_2) & s_1 \neq s_1 \oplus_S s_2 = s_2 \end{cases}$$

Lexicographic product for Bi-semigroups

Suppose that

$$\mathbb{AS}(S,\ \oplus_S) \wedge \mathbb{CM}(S,\ \oplus_S) \wedge \mathbb{SL}(S,\ \oplus_S) \wedge \mathbb{AS}(T,\ \oplus_T).$$

Let

$$(S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T) \equiv (S \times T,\ \oplus_S \vec{\times} \oplus_T,\ \otimes_S \times \otimes_T)$$

Examples

$\oplus = \min \vec{\times} \max$, $\otimes = + \times \min$

$$
\begin{array}{rcl}
(3, 10) \otimes ((17, 21) \oplus (11, 4)) & = & (3, 10) \otimes (11, 4) \\
& = & (14, 4) \\
\\
((3, 10) \otimes (17, 21)) \oplus ((3, 10) \otimes (11, 4)) & = & (20, 10) \oplus (14, 4) \\
& = & (14, 4)
\end{array}
$$

$\oplus = \max \vec{\times} \min$, $\otimes = \min \times +$

$$
\begin{array}{rcl}
(3, 10) \otimes ((17, 21) \oplus (11, 4)) & = & (3, 10) \otimes (17, 21) \\
& = & (3, 31) \\
\\
((3, 10) \otimes (17, 21)) \oplus ((3, 10) \otimes (11, 4)) & = & (3, 31) \oplus (3, 14) \\
& = & (3, 14)
\end{array}
$$

Distributivity?

Theorem: If $\oplus_S$ is commutative and selective, then

$$\mathbb{LD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T))\ \Leftrightarrow$$
$$\mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T))$$

o

$$\mathbb{RD}((S,\ \oplus_S,\ \otimes_S)\ \vec{\times}\ (T,\ \oplus_T,\ \otimes_T))\ \Leftrightarrow$$
$$\mathbb{RD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{RD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{RC}(S,\ \otimes_S) \vee \mathbb{RK}(T,\ \otimes_T))$$

Left and Right Cancellative

$$\begin{array}{lcl} \mathbb{LC}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b \\ \mathbb{RC}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ a \bullet c = b \bullet c \Rightarrow a = b \end{array}$$

Left and Right Constant

$$\begin{array}{lcl} \mathbb{LK}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \\ \mathbb{RK}(X,\ \bullet) & \equiv & \forall a, b, c \in X,\ a \bullet c = b \bullet c \end{array}$$

Why bisemigroups?

But wait! How could any semiring satisfy either of these properties?

$$\begin{array}{rcl} \mathbb{LC}(X, \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b \\ \mathbb{LK}(X, \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \end{array}$$

- For $\mathbb{LC}$, note that we always have $\overline{0} \otimes a = \overline{0} \otimes b$, so $\mathbb{LC}$ could only hold when $S = \{\overline{0}\}$.
- For $\mathbb{LK}$, let $a = \overline{1}$ and $b = \overline{0}$ and $\mathbb{LK}$ leads to the conclusion that every c is equal to $\overline{0}$ (again!).

Normally we will add a zero and/or a one as the last step(s) of constructing a semiring. Alternatively, we might want to complicate our properties so that things work for semirings. A design trade-off!

Proof of $\Leftarrow$ for $\mathbb{LD}$

Assume

$$\begin{array}{ll} (1) & \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \\ (2) & \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \\ (3) & \mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T) \\ (4) & \mathbb{IP}(S,\ \oplus_S). \end{array}$$

Let $\oplus \equiv \oplus_S \vec{\times} \oplus_T$ and $\otimes \equiv \otimes_S \times \otimes_T$. Suppose

$$(s_1, t_1),\ (s_2, t_2),\ (s_3, t_3) \in S \times T.$$

We want to show that

$$\begin{array}{lll} \text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\ & = & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\ & \equiv & \text{rhs} \end{array}$$

Proof of $\Leftarrow$ for $\mathbb{LD}$

We have

$$
\begin{array}{lcl}
\text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\
 & = & (s_1,\ t_1) \otimes (s_2 \oplus_S s_3,\ t_{\text{lhs}}) \\
 & = & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_1 \otimes_T t_{\text{lhs}})
\end{array}
$$

$$
\begin{array}{lcl}
\text{rhs} & \equiv & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\
 & = & (s_1 \otimes_S s_2,\ t_1 \otimes_T t_2) \oplus (s_1 \otimes_S s_3,\ t_1 \otimes_T t_3) \\
 & = & ((s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3),\ t_{\text{rhs}}) \\
 & =_{(1)} & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_{\text{rhs}})
\end{array}
$$

where t_{lhs} and t_{rhs} are determined by the appropriate case in the definition of $\oplus$. Finally, note that

$$\text{lhs} = \text{rhs} \Leftrightarrow t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}.$$

Proof by cases on $s_2 \oplus_S s_3$

Case 1 : $s_2 = s_2 \oplus_S s_3 = s_3$. Then $t_{\mathrm{lhs}} = t_2 \oplus_T t_3$ and

$$t_1 \otimes_T t_{\mathrm{lhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) =_{(2)} (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3).$$

Since $s_2 = s_3$ we have $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$ and

$$s_1 \otimes_S s_2 =_{(4)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3) =_{(4)} s_1 \otimes_S s_3.$$

Therefore,

$$t_{\mathrm{rhs}} = (t_1 \otimes_T t_2) \oplus (t_1 \otimes_T t_3) = t_1 \otimes_T t_{\mathrm{lhs}}.$$

Case 2 : $s_2 = s_2 \oplus_S s_3 \neq s_3$. Then $t_{\mathrm{lhs}} = t_2$ and

$$t_1 \otimes_T t_{\mathrm{lhs}} = t_1 \otimes_T t_2.$$

Since $s_2 = s_2 \oplus_S s_3$ we have

$$s_1 \otimes_S s_2 = s_1 \otimes_S (s_2 \oplus_S s_3) =_{(1)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3).$$

Case 2.1 $s_1 \otimes_S s_2 \neq s_1 \otimes_S s_3$. Then $t_{\text{rhs}} = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\text{lhs}}$.
Case 2.2 $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$. Then

$$t_{\text{rhs}} = (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) =_{(2)} t_1 \otimes_T (t_2 \oplus_T t_3)$$

We need to consider two subcases.
Case 2.2.1: Assume $\mathbb{LC}(S, \otimes_S)$. But $s_1 \otimes_S s_2 = s_1 \otimes_S s_3 \Rightarrow s_2 = s_3$, which is a contradiction.
Case 2.2.2 : Assume $\mathbb{LK}(T, \otimes_T)$. In this case we know

$$\forall a, b \in X,\ t_1 \otimes_T a = t_1 \otimes_T b.$$

Letting $a = t_2 \oplus_T t_3$ and $b = t_2$ we have

$$t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\text{lhs}}.$$

Case 3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to Case 2.

Other direction, $\Rightarrow$

Prove this:

$$\neg\mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \vee \neg\mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \vee (\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$$
$$\Rightarrow\ \neg\mathbb{LD}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)).$$

Case 1: $\neg\mathbb{LD}(S,\ \oplus_S,\ \otimes_S)$. That is

$$\exists a, b, c \in S,\ a \otimes_S (b \oplus_S c) \neq (a \otimes_S b) \oplus_S (a \otimes_S c).$$

Pick any $t \in T$. Then for some $t_1, t_2, t_3 \in T$ we have

$$\begin{array}{cl} & (a,\ t) \otimes ((b,\ t) \oplus (c,\ t)) \\ = & (a,\ t) \otimes (b \oplus_S c,\ t_1) \\ = & (a, \otimes_S (b \oplus_S c),\ t_2) \\ \neq & ((a \otimes_S b) \oplus_S (a \otimes_S c),\ t_3) \\ = & (a \otimes_S b,\ t \otimes_T t) \oplus (a \otimes_S c,\ t \otimes_T t) \\ = & ((a,\ t) \otimes (b,\ t)) \oplus ((a,\ t) \otimes (c,\ t)) \end{array}$$

Case 2: $\neg\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$. Similar.

Case 3: $(\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$. That is

$$\exists a, b, c \in S,\ c \otimes_S a = c \otimes_S b \wedge a \neq b$$

and

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

Since $\oplus_S$ is selective and $a \neq b$, we have $a = a \oplus_S b$ or $b = a \oplus_S b$. Assume without loss of generality that $a = a \oplus_S b \neq b$. Suppose that $t_1, t_2, t_3 \in T$. Then

$$\begin{array}{rcl} \text{lhs} & \equiv & (c,\ t_1) \otimes ((a,\ t_2) \oplus (b,\ t_3)) \\ & = & (c,\ t_1) \otimes (a,\ t_2) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \end{array}$$

$$\begin{array}{rcl} \text{rhs} & \equiv & ((c,\ t_1) \otimes (a,\ t_2)) \oplus ((c,\ t_1) \otimes (b,\ t_3)) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \oplus (c \otimes_S b,\ t_1 \otimes_T t_3) \\ & = & (c \otimes_S a,\ (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \end{array}$$

Our job now is to select t_1, t_2, t_3 so that

$$t_{\mathrm{lhs}} \equiv t_1 \otimes_T t_2 \neq (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) \equiv t_{\mathrm{rhs}}.$$

We don't have very much to work with! Only

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

In addition, we can assume $\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$ (otherwise, use Case 2!), so

$$t_{\mathrm{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3).$$

We need to select t_1, t_2, t_3 so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq t_1 \otimes_T (t_2 \oplus_T t_3) \equiv t_{\text{rhs}}.$$

Case 3.1: $z \otimes_T x = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = y$, and $t_3 = x$ we have

$$t_{\text{lhs}} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.2: $z \otimes_T y = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T y = z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.3: $z \otimes_T x \neq z \otimes_T (x \oplus_T y) \neq z \otimes_T y$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

□

Today

- Widest shortest paths
- Solving some matrix equations
- Counting to infinity, as does RIP

Widest shortest paths

$$\begin{array}{rcl} \text{wsp} & \equiv & \text{AddZero}(\infty_2,\ (\mathbb{N},\ \mathsf{min},\ +)\ \vec{\times}\ \text{AddOne}(\infty_1,\ (\mathbb{N},\ \mathsf{max},\ \mathsf{min}))) \\ & = & ((\mathbb{N} \times (\mathbb{N} \uplus \{\infty_1\})) \uplus \{\infty_2\},\ \oplus,\ \otimes,\ \text{inr}(\infty_2),\ \text{inl}(0,\ \text{inr}(\infty_1))) \end{array}$$

where

$$\begin{array}{rcl} \oplus & = & (\mathsf{min}\ \vec{\times}\ \mathsf{max}^{\text{an}}_{\infty_1})^{\text{id}}_{\infty_2} \\ \otimes & = & (+ \times \mathsf{min}^{\text{id}}_{\infty_1})^{\text{an}}_{\infty_2} \end{array}$$

Example

$$
\begin{aligned}
& \mathrm{inl}(3, \mathrm{inl}(10)) \otimes (\mathrm{inl}(17, \mathrm{inl}(21)) \oplus \mathrm{inl}(11, \mathrm{inl}(4))) \\
= \;& \mathrm{inl}(3, \mathrm{inl}(10)) \otimes \mathrm{inl}(11, \mathrm{inl}(4)) \\
= \;& \mathrm{inl}(14, \mathrm{inl}(4))
\end{aligned}
$$

$$
\begin{aligned}
& (\mathrm{inl}(3, \mathrm{inl}(10)) \otimes \mathrm{inl}(17, \mathrm{inl}(21))) \oplus (\mathrm{inl}(3, \mathrm{inl}(10)) \otimes \mathrm{inl}(11, \mathrm{inl}(4))) \\
= \;& \mathrm{inl}(20, \mathrm{inl}(10)) \oplus \mathrm{inl}(14, \mathrm{inl}(4)) \\
= \;& \mathrm{inl}(14, \mathrm{inl}(4))
\end{aligned}
$$

But is wsp a semiring?

Turn the cranks!

Turning the crank for $\mathbb{LD}$:

$$\mathbb{LD}(\text{AddZero}(\infty_2, (\mathbb{N}, \min, +) \ \vec{\times}\ \text{AddOne}(\infty_1, (\mathbb{N}, \max, \min))))$$

$$\Leftrightarrow \quad \mathbb{LD}((\mathbb{N}, \min, +) \ \vec{\times}\ \text{AddOne}(\infty_1, (\mathbb{N}, \max, \min)))$$

$$\Leftrightarrow \quad \mathbb{LD}(\mathbb{N}, \min, +) \wedge \mathbb{LD}(\text{AddOne}(\infty_1, (\mathbb{N}, \max, \min))) \wedge (\mathbb{LC}(\mathbb{N}, +) \vee \mathbb{LK}(\text{AddID}(\infty_1, (\mathbb{N}, \min)))$$

$$\Leftrightarrow \quad \mathbb{TRUE} \wedge (\mathbb{LD}(\mathbb{N}, \max, \min) \wedge \mathbb{RAB}(\mathbb{N}, \max, \min) \wedge \mathbb{IP}(\mathbb{N}, \max)) \wedge (\mathbb{TRUE} \vee \mathbb{LK}(\text{AddID}(\infty_1, (\mathbb{N}, \min)))$$

$$\Leftrightarrow \quad \mathbb{TRUE} \wedge (\mathbb{TRUE} \wedge \mathbb{TRUE} \wedge \mathbb{TRUE}) \wedge (\mathbb{TRUE} \vee \mathbb{LK}(\text{AddID}(\infty_1, (\mathbb{N}, \min)))$$

$$\Leftrightarrow \quad \mathbb{TRUE}$$

Solving (some) equations

Theorem 6.1

If $\mathbf{A}$ is q-stable, then $\mathbf{A}^*$ solves the equations

$$\mathbf{L} = \mathbf{A}\mathbf{L} \oplus \mathbf{I}$$

and

$$\mathbf{R} = \mathbf{R}\mathbf{A} \oplus \mathbf{I}.$$

For example, to show $\mathbf{L} = \mathbf{A}^*$ solves the first equation:

$$\begin{aligned}
\mathbf{A}^* &= \mathbf{A}^{(q)} \\
&= \mathbf{A}^{(q+1)} \\
&= \mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^{2} \oplus \mathbf{A} \oplus \mathbf{I} \\
&= \mathbf{A}(\mathbf{A}^{q} \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I}) \oplus \mathbf{I} \\
&= \mathbf{A}\mathbf{A}^{(q)} \oplus \mathbf{I} \\
&= \mathbf{A}\mathbf{A}^{*} \oplus \mathbf{I}
\end{aligned}$$

Note that if we replace the assumption "$\mathbf{A}$ is q-stable" with "$\mathbf{A}^*$ exists," then we require that $\otimes$ distributes over <u>infinite</u> sums.

A more general result

Theorem Left-Right

If $\mathbf{A}$ is q-stable, then $\mathbf{L} = \mathbf{A}^*\mathbf{B}$ solves the equation

$$\mathbf{L} = \mathbf{A}\mathbf{L} \oplus \mathbf{B}$$

and $\mathbf{R} = \mathbf{B}\mathbf{A}^*$ solves

$$\mathbf{R} = \mathbf{R}\mathbf{A} \oplus \mathbf{B}.$$

For the first equation:

$$\begin{aligned}
\mathbf{A}^*\mathbf{B} &= \mathbf{A}^{(q)}\mathbf{B} \\
&= \mathbf{A}^{(q+1)}\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \\
&= (\mathbf{A}^{q+1} \oplus \mathbf{A}^{q} \oplus \ldots \oplus \mathbf{A}^2 \oplus \mathbf{A})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^{q} \oplus \mathbf{A}^{q-1} \oplus \ldots \oplus \mathbf{A} \oplus \mathbf{I})\mathbf{B} \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^{(q)}\mathbf{B}) \oplus \mathbf{B} \\
&= \mathbf{A}(\mathbf{A}^*\mathbf{B}) \oplus \mathbf{B}
\end{aligned}$$

The "best" solution

Suppose $\mathbf{Y}$ is a matrix such that

$$\mathbf{Y} = \mathbf{AY} \oplus \mathbf{I}$$

$$\begin{aligned}
\mathbf{Y} &= \mathbf{AY} \oplus \mathbf{I} \\
&= \mathbf{A}^1\mathbf{Y} \oplus \mathbf{A}^{(0)} \\
&= \mathbf{A}((\mathbf{AY} \oplus \mathbf{I})) \oplus \mathbf{I} \\
&= \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A} \oplus \mathbf{I} \\
&= \mathbf{A}^2\mathbf{Y} \oplus \mathbf{A}^{(1)} \\
\vdots & \quad \vdots \quad \vdots \\
&= \mathbf{A}^{k+1}\mathbf{Y} \oplus \mathbf{A}^{(k)}
\end{aligned}$$

If $\mathbf{A}$ is q-stable and $q < k$, then

$$\mathbf{Y} = \mathbf{A}^k\mathbf{Y} \oplus \mathbf{A}^*$$

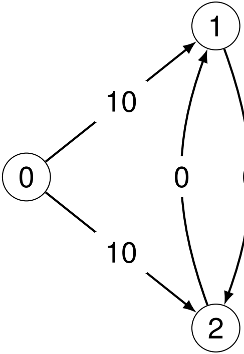
$$\mathbf{Y} \trianglelefteq_{\oplus}^{L} \mathbf{A}^*$$

and if $\oplus$ is idempotent, then

$$\mathbf{Y} \leq_{\oplus}^{L} \mathbf{A}^*$$

So $\mathbf{A}^*$ is the largest solution. What does this mean in terms of the sp semiring?

Example with zero weighted cycles using sp semiring



$$\mathbf{A} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left[\begin{array}{ccc} \infty & 10 & 10 \\ \infty & \infty & 0 \\ \infty & 0 & \infty \end{array}\right] \end{array}$$

$\mathbf{A}^*$ ($= \mathbf{A} \oplus \mathbf{I}$ in this case) solves

$$\mathbf{X} = \mathbf{X}\mathbf{A} \oplus \mathbf{I}.$$

But so does this (dishonest) matrix!

$$\mathbf{F} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left[\begin{array}{ccc} 0 & 9 & 9 \\ \infty & 0 & 0 \\ \infty & 0 & 0 \end{array}\right] \end{array}$$

For example :

$$\begin{aligned} & (\mathbf{F}\mathbf{A} \oplus \mathbf{I})(0, 1) \\ = & \min_{q \in \{0,1,2\}} \mathbf{F}(0, q) + \mathbf{A}(q, 1) \\ = & \min(0 + 10, 9 + \infty, 9 + 0) \\ = & 9 \\ = & \mathbf{F}(0, 1) \end{aligned}$$

Recall our basic iterative algorithm

$$\begin{aligned} \mathbf{A}^{\langle 0 \rangle} &= \mathbf{I} \\ \mathbf{A}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

A closer look ...

$$\begin{aligned} \mathbf{A}^{\langle k+1 \rangle}(i,j) &= \mathbf{I}(i,j) \oplus \bigoplus_{u} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \\ &= \mathbf{I}(i,j) \oplus \bigoplus_{(i,u) \in E} \mathbf{A}(i,u)\mathbf{A}^{\langle k \rangle}(u,j) \end{aligned}$$

This is the basis of distributed Bellman-Ford algorithms (as in RIP and BGP) — a node i computes routes to a destination j by applying its link weights to the routes learned from its immediate neighbors. It then makes these routes available to its neighbors and the process continues...

What if we start iteration in an arbitrary state $\mathbf{M}$?

In a distributed environment the topology (captured here by **A**) can change and the state of the computation can start in an arbitrary state (with respect to a new **A**).

$$\begin{aligned} \mathbf{A}_{\mathbf{M}}^{\langle 0 \rangle} &= \mathbf{M} \\ \mathbf{A}_{\mathbf{M}}^{\langle k+1 \rangle} &= \mathbf{A}\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} \oplus \mathbf{I} \end{aligned}$$

Theorem

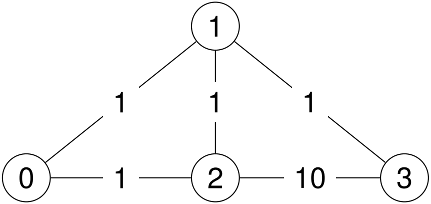
For $1 \leq k$,

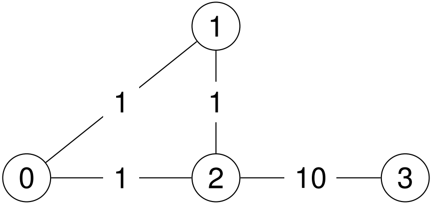
$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^k\mathbf{M} \oplus \mathbf{A}^{(k-1)}$$

If **A** is q-stable and $q < k$, then

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle} = \mathbf{A}^k\mathbf{M} \oplus \mathbf{A}^*$$

RIP-like example — counting to convergence (1)





Adjacency matrix $\mathbf{A}_1$

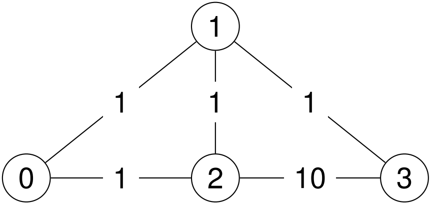
$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
\infty & 1 & 1 & \infty \\
1 & \infty & 1 & 1 \\
1 & 1 & \infty & 10 \\
\infty & 1 & 10 & \infty
\end{array}\right]
\end{array}
$$

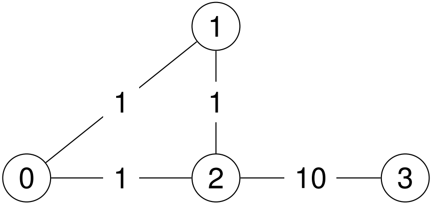
Adjacency matrix $\mathbf{A}_2$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccc}
\infty & 1 & 1 & \infty \\
1 & \infty & 1 & \infty \\
1 & 1 & \infty & 10 \\
\infty & \infty & 10 & \infty
\end{array}\right]
\end{array}
$$

See RFC 1058.

RIP-like example — counting to convergence (2)





The solution $\mathbf{A}_1^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

The solution $\mathbf{A}_2^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 11 \\ 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 10 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

RIP-like example — counting to convergence (3)

The scenario: we arrived at $\mathbf{A}_1^*$, but then links $\{(1,3),\ (3,1)\}$ fail. So we start iterating using the new matrix $\mathbf{A}_2$.

Let $\mathbf{B}_K$ represent ${\mathbf{A}_2}_{\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to convergence (4)

$$\mathbf{B}_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 4 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_4 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 5 \\ 1 & 1 & 0 & 5 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{B}_5 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 6 \\ 1 & 0 & 1 & 6 \\ 1 & 1 & 0 & 6 \\ 11 & 11 & 10 & 0 \end{bmatrix} \end{array}$$

RIP-like example — counting to convergence (5)

$$\mathbf{B}_6 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 7 \\ 1 & 1 & 0 & 1 & 7 \\ 2 & 1 & 1 & 0 & 7 \\ 3 & 2 & 1 & 2 & 0 \end{array}$$

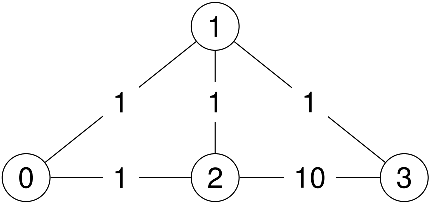
$$\mathbf{B}_7 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 8 \\ 1 & 1 & 0 & 1 & 8 \\ 2 & 1 & 1 & 0 & 8 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

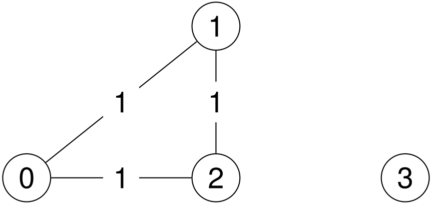
$$\mathbf{B}_8 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 9 \\ 1 & 1 & 0 & 1 & 9 \\ 2 & 1 & 1 & 0 & 9 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

$$\mathbf{B}_9 = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 10 \\ 1 & 1 & 0 & 1 & 10 \\ 2 & 1 & 1 & 0 & 10 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

$$\mathbf{B}_{10} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 11 \\ 1 & 1 & 0 & 1 & 11 \\ 2 & 1 & 1 & 0 & 10 \\ 3 & 11 & 11 & 10 & 0 \end{array}$$

RIP-like example — counting to infinity (1)





The solution $\mathbf{A}_1^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{array}\right] \end{array}$$

The solution $\mathbf{A}_3^*$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} 0 & 1 & 1 & \infty \\ 1 & 0 & 1 & \infty \\ 1 & 1 & 0 & \infty \\ \infty & \infty & \infty & 0 \end{array}\right] \end{array}$$

Now let $\mathbf{B}_K$ represent $\mathbf{A_3}_{\mathbf{M}}^{\langle k \rangle}$, where $\mathbf{M} = \mathbf{A}_1^*$.

RIP-like example — counting to infinity (2)

$$
\mathbf{B}_0 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}
$$

$$
\mathbf{B}_1 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 2 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}
$$

$$
\mathbf{B}_2 = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 3 \\ 1 & 1 & 0 & 3 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}
$$

$$
\vdots \quad \vdots \quad \vdots
$$

$$
\mathbf{B}_{376} = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 377 \\ 1 & 0 & 1 & 377 \\ 1 & 1 & 0 & 377 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}
$$

$$
\vdots \quad \vdots \quad \vdots
$$

$$
\mathbf{B}_{998} = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 999 \\ 1 & 0 & 1 & 999 \\ 1 & 1 & 0 & 999 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}
$$

$$
\vdots \quad \vdots \quad \vdots
$$

RIP-like example — What's going on?

Recall

$$\mathbf{A}_{\mathbf{M}}^{\langle k \rangle}(i,\ j) = \mathbf{A}^k\mathbf{M}(i,\ j) \oplus \mathbf{A}^*(i,\ j)$$

- $\mathbf{A}^*$(i, j) may be arrived at very quickly
- but $\mathbf{A}^k\mathbf{M}(i,\ j)$ may be better until a very large value of k is reached (counting to convergence)
- or it may always be better (counting to infinity).

Solutions?

- RIP: $\infty = 16$
- In the next lecture we will explore various ways of adding paths to metrics and eliminating those paths with loops ....