Recall

Algebraic terms over a signature $\Sigma$

Fix a countably infinite set of variables $x, y, z, \ldots$

Terms:

$$t ::= x \mid f(t_1, \ldots, t_n)$$

N.B. When $n = 0$, often write $f()$ just as $f$.
Substitution

The substituted term $t'[t/x]$ is defined by recursion on the structure of the term $t'$:

- if $t'$ is a variable, then
  $$y[t/x] = \begin{cases} t & \text{if } y = x \\ y & \text{otherwise} \end{cases}$$

- if $t'$ is a compound term, then
  $$f(t'_1, \ldots, t'_m)[t/x] = f(t'_1[t/x], \ldots, t'_m[t/x])$$
Simultaneous substitution

t'[t/x] is a special case of

t'[t_1/x_1, ..., t_n/x_n] (where x_1, ..., x_n are distinct)

defined by:

y [t_1/x_1, ..., t_n/x_n] = \begin{cases} t_i & \text{if } y = x_i; \\ y & \text{if } y \notin \{x_1, ..., x_n\} \end{cases}

f(t_1', ..., t_m') [t_i/x_1, ...] = f(t_1' [t_i/x_1, ...], ..., t_m' [t_i/x_1, ...])
Recall: Typing judgement over a signature $\Sigma$ is inductively generated by the rules...

\[
\frac{(x : S) \in \Gamma}{\Gamma \vdash_x x : S}
\]

\[
\Gamma \vdash_{\Sigma} t_1 : S_1, \ldots, \Gamma \vdash_{\Sigma} t_n : S_n \quad (f : [S_1, \ldots, S_n] \to S) \in \Sigma
\]

\[
\Gamma \vdash_{\Sigma} f(t_1, \ldots, t_n) : S
\]
Lemma
If $\Gamma \vdash t_i : S_i, \ldots, \Gamma \vdash t_n : S_n$ and $x_i : S_i, \ldots, x_n : S_n \vdash t' : S'$, then $\Gamma \vdash t'[t_i/x_i, \ldots, t_n/x_n] : S'$

Proof by induction on the structure of $t'$.

Substitution lemma
For any structure for an alg. sig. in a cartesian category:
$[[ t'[t_i/x_i, \ldots, t_n/x_n] ]] = [[t']] \circ [[[[t_1], \ldots, [t_n]]]]$

Proof by induction on the structure of $t'$. 
Recall

Structures

A structure for an algebraic signature $\Sigma$ in a cartesian category $\mathcal{C}$ allows us to interpret each valid typing judgement $\Gamma \vdash t : S$ as a $\mathcal{C}$-morphism

$\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \to \llbracket S \rrbracket$

$\llbracket x \rrbracket = \pi_i : S_i \times \cdots \times S_n \to S_i \quad \text{if} \quad x = x_i$

$\llbracket f(t_1, \ldots, t_n) \rrbracket = \llbracket f \rrbracket \circ \langle \llbracket t_1 \rrbracket, \ldots, \llbracket t_n \rrbracket \rangle$
Equations over a signature $\Sigma$ take the form $\Gamma \vdash_\Sigma t = t' : S$

where $\Gamma \vdash_\Sigma t : S$ and $\Gamma \vdash_\Sigma t' : S$.

Algebraic theory = \{ algebraic signature + set of equations (the theory's axioms) \}

E.g. alg. theory of monoids has equations

\[
x : * , y : * , z : * \vdash m(x, m(y, z)) = m(m(x, y), z) : *
\]
\[
x : * \vdash m(u(), x) = x : *
\]
\[
x : * \vdash m(x, u()) = x : *
\]
Example: an algebraic theory of lists

sorts: \( V \) (values) \( L \) (lists of values)

operation symbols:

\( \text{nil} : [] \to L \) \( \text{hd} : [L] \to V \)

\( \text{cons} : [V, L] \to L \) \( \text{tl} : [L] \to L \)

\( \text{append} : [L, L] \to L \)
Example: an algebraic theory of lists

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operation symbols:
- \( \text{nil} : [] \to L \)
- \( \text{hd} : [L] \to V \)
- \( \text{tl} : [L] \to L \)
- \( \text{cons} : [V, L] \to L \)
- \( \text{apnd} : [L, L] \to L \)

axioms:
- \( x : V, l : L \vdash \text{hd} (\text{cons}(x, l)) = x : V \)
- \( x : V, l : L \vdash \text{tl} (\text{cons}(x, l)) = l : L \)
- \( l : L \vdash \text{apnd}(\text{nil}, l) = l : L \)
- \( x : V, l : L, l' : L \vdash \text{apnd}(\text{cons}(x, l), l') = \text{cons}(x, \text{apnd}(l, l')) : L \)
The theorems of an algebraic theory consist of all the equations derivable from its axioms using the following rules...

\[
\begin{align*}
\frac{\Gamma, t : S}{\Gamma, t = t : S} \\
\frac{\Gamma, t = t' : S}{\Gamma, t' = t : S} \\
\frac{\Gamma, t = t' : S \Gamma, t = t'' : S}{\Gamma, t = t'' : S}
\end{align*}
\]

\[
\begin{align*}
\Gamma, t_1 = t'_1 : S_1 \quad \ldots \quad \Gamma, t_n = t'_n : S_n \\
x_1 : S_1, \ldots, x_n : S_n \quad \Gamma, t = t' : S \\
\Gamma, \frac{t[t_1/x_1, \ldots, t_n/x_n]}{\Sigma} = t'[t'_1/x_1, \ldots, t'_n/x_n] : S
\end{align*}
\]
Satisfaction

A structure in a cartesian category for an algebraic signature satisfies an equation \( \exists t \in S \text { s.t. } \llcorner t \llcorner = \llcorner t' \llcorner : \llcorner 1 \llcorner \to \llcorner S \llcorner \) if

Soundness Theorem. If a structure satisfies all the axioms of an algebraic theory, then it satisfies all its theorems.

Proof. Just have to check that satisfaction is closed under the rules of equational logic. For the substitution rule, use the substitution lemma.
Algebras

An algebra for an algebraic theory \( T \) in a cartesian category \( \mathcal{C} \) is a structure that satisfies all the axioms of \( T \).

[There's a category of \( T \)-algebras in \( \mathcal{C} \), the morphism of which are algebra homomorphisms (definition omitted).]
The internal language of a cartesian category $\mathcal{C}$ is the algebraic signature with:

- one sort for each $\mathcal{C}$-object
- one operation symbol $f : [x_1, \ldots, x_n] \to x$ for each non-empty list $[x_1, \ldots, x_n, x]$ of $\mathcal{C}$-objects and each $\mathcal{C}$-morphism $f : x_1 \times \cdots \times x_n \to x$
The internal language of a cartesian category \( \mathcal{C} \) is the algebraic signature with:

- one sort for each \( \mathcal{C} \)-object
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\[
\begin{align*}
\mathcal{E} x f &= x \\
\mathcal{E} f &= f
\end{align*}
\]

Terms & equations over this signature allow us to describe properties in \( \mathcal{C} \), for example...
Proposition

1. \( X \in C \) is a terminal object in \( C \) iff there is some \( C \)-morphism \( t : 1 \rightarrow X \) satisfying (in the internal lang. of \( C \))
   \[ x : X \vdash x = t() : X \]
   "This has just one element"

2. \( X \leftarrow Z \rightarrow Y \) is a product in \( C \) iff there is a \( C \)-morphism \( r : X \times Y \rightarrow Z \) satisfying
   \[ x : X, y : Y \vdash p(r(x, y)) = x : X \]
   \[ x : X, y : Y \vdash q(r(x, y)) = y : Y \]
   \[ z : Z \vdash z = r(p(z), q(z)) : Z \]
Theories as categories

From an algebraic theory $\mathcal{T}$, can construct a cartesian category $\mathcal{C}_T$:

- objects are typing contexts $\Gamma = x_1 : S_1 , \ldots , x_n : S_n$

- morphisms $\Gamma \to \Gamma'$ are equivalence classes $[t_1' , \ldots , t_m']$ where
  $\Gamma' = x_1' : S_1' , \ldots , x_m' : S_m'$ and $\Gamma \vdash t_i' : S_i$ (i=1,..m)
  for equiv. relation given by theorem of $\mathcal{T}$

- composition given by substitution, identities given by variables
Theories as categories

From an algebraic theory $T$, one construct a cartesian category $\mathcal{C}_T$ — see Section 4.2 of [Pilts, Categorical Logic].

There's a structure for $T$ in $\mathcal{C}_T$ that satisfies all its axioms.

and $\Gamma \vdash t = t' : S$ is a theorem of $T$ iff it's satisfied by this structure.

Hence cartesian categories are complete for equational logic.