

Category-theoretic properties

"Any two isomorphic objects in a category have the same category-theoretic properties."

instead of formalizing the "language & logic of category theory", we'll just look at examples of category-theoretic properties.

Here's our first one...

Terminal objects

An object $T \in \mathcal{C}$ of a category \mathcal{C} is **terminal** if for all $X \in \mathcal{C}$, there is a unique morphism $X \rightarrow T$ (we'll write $\langle \rangle_x$, or just $\langle \rangle$ for this morphism)

Theorem In a category \mathcal{C} :

(a) if T is terminal & $T \cong T'$, then T' is terminal

(b) if T & T' are both terminal, then $T \cong T'$ (and there is only one isomorphism between T & T')

terminal objects are unique up to unique isomorphism

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Theorem In a category \mathcal{C} :

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- (b) if T & T' are both terminal, then $T \cong T'$
(and ...)

Proof ...

Examples of terminal objects

- In Set : any one-element set
- Any one-element set has a unique pre-order & this makes it terminal in Pre
- Ditto for Mon .
- A pre-ordered set (P, \leq) , regarded as a category, has a terminal object iff it has a **greatest element**: $(\forall x \in P) x \leq T$
- When does a monoid $(M, \cdot, 1)$, regarded as a category, have a terminal object?

The opposite of a category \mathcal{C}

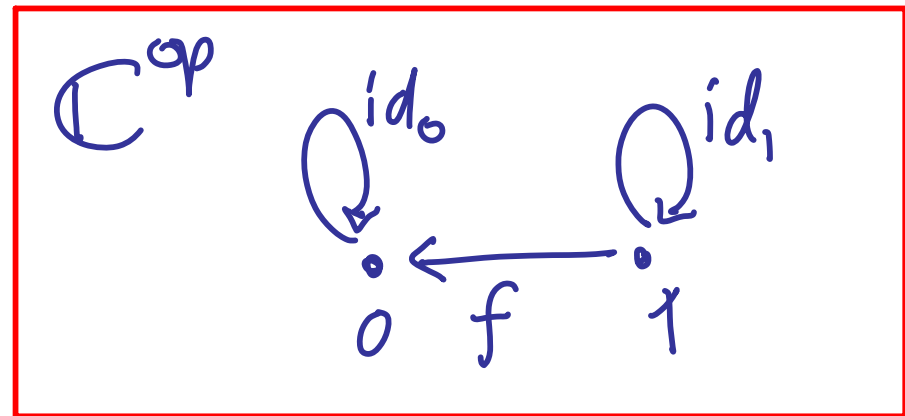
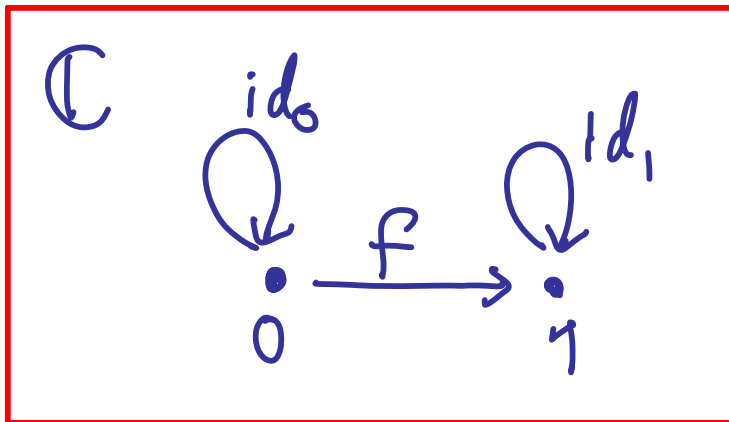
is the category \mathcal{C}^{op} defined by

- $\text{Obj } \mathcal{C}^{\text{op}} \triangleq \text{Obj } \mathcal{C}$
- $\mathcal{C}^{\text{op}}(X, Y) \triangleq \mathcal{C}(Y, X)$ for all objects X, Y

same objects

same morphisms, but with direction reversed, that is, dom & cod swapped

E.g.



The opposite of a category \mathcal{C}

is the category \mathcal{C}^{op} defined by

- $\text{Obj } \mathcal{C}^{\text{op}} \triangleq \text{Obj } \mathcal{C}$
- $\mathcal{C}^{\text{op}}(X, Y) \triangleq \mathcal{C}(Y, X)$ for all objects X, Y
- identity morphism on $X \in \text{Obj } \mathcal{C}^{\text{op}}$ is id_X , the identity on $X \in \text{Obj } \mathcal{C}$
- the composition of $f \in \mathcal{C}^{\text{op}}(X, Y)$ & $g \in \mathcal{C}^{\text{op}}(Y, Z)$ is given by composition $f \circ g \in \mathcal{C}(Z, X)$ in \mathcal{C}
 $g \circ_{\mathcal{C}^{\text{op}}} f \triangleq f \circ_{\mathcal{C}} g$

(associativity & unity props hold, because they do in \mathcal{C})

Principle of Duality

Whenever we { define a concept in terms of
{ prove a theorem
Commutative diagrams, we obtain another
{ concept, called its **dual**, by reversing
{ theorem, by reversing
the direction of morphisms throughout (i.e.
by replacing \mathbb{C} by \mathbb{C}^{op}).

For example...

Initial object

is the dual notion to "terminal object"

An object $I \in \mathcal{C}$ of a category \mathcal{C} is **initial** if for all $X \in \mathcal{C}$, there is a unique morphism $I \rightarrow X$
(we'll write $[\]_X$, or just $[\]$ for this morphism)

By duality, we have that initial objects are unique up to iso and that any object isomorphic to an initial object is itself initial.

NB "isomorphism" is a self-dual concept

Examples of initial objects

- The empty set is initial in **Set**
- Any one-element monoid (has uniquely determined monoid operation & unit) is initial in **Mon** (why?)
→ (so initial & terminal objects coincide in **Mon**)

an object that's both initial & terminal is sometimes called a zero object

Example: free monoids as initial objects

(relevant to automata & formal languages)

Free monoid on a set $\Sigma \in \text{Set}$:

$(\text{List}(\Sigma), @, \text{nil})$

Set of finite lists
of elements of Σ

empty
list

list
concatenation:
 $\text{nil} @ l' = l'$
 $(a :: l) @ l' =$
 $a :: (l @ l')$

Example: free monoids as initial objects

Free monoid on a set $\Sigma \in \text{Set}$:

$i_\Sigma : \Sigma \rightarrow \text{List}(\Sigma)$ in Set

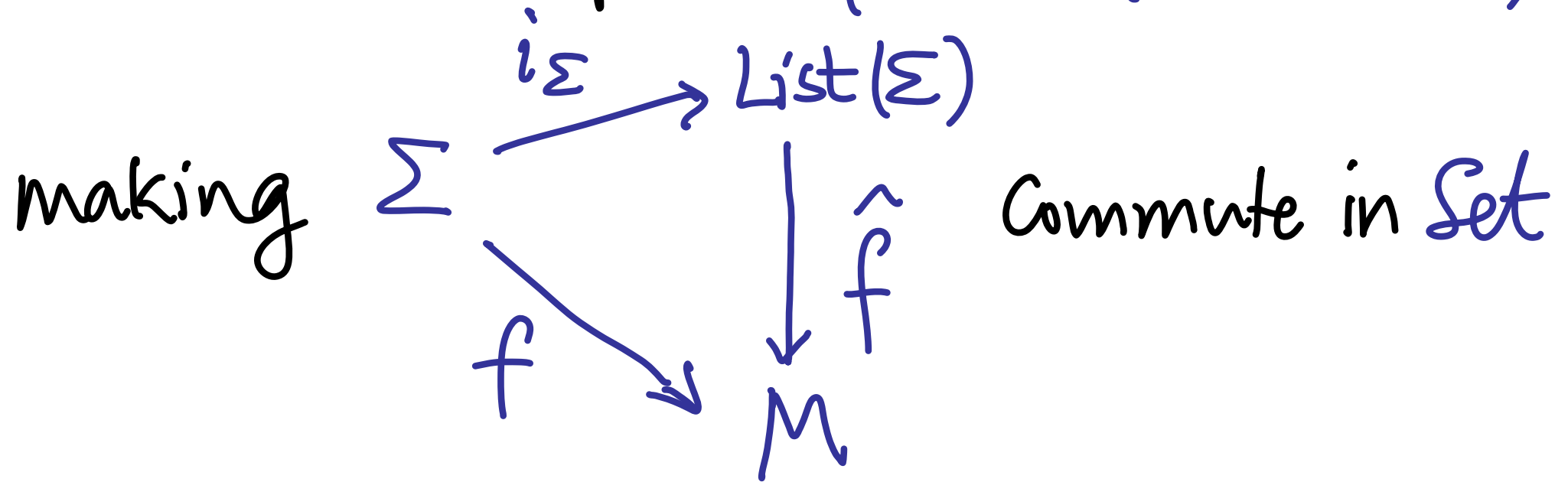
$a \mapsto [a]$ where $[a] \triangleq a :: \text{nil}$

i_Σ sends element $a \in \Sigma$ to corr. list of length 1

It has the following "universal property"...

Example: free monoids as initial objects

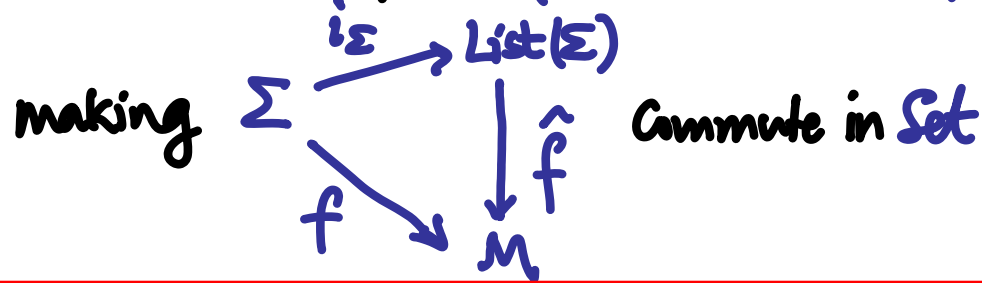
Theorem Given $\Sigma \in \text{Set}$, $(M, \cdot, e) \in \text{Mon}$ and $f \in \text{Set}(\Sigma, M)$, there is a unique monoid homomorphism $\hat{f} \in \text{Mon}(\text{List}(\Sigma), M)$



Proof ...

Example: free monoids as initial objects

Theorem Given $\Sigma \in \text{Set}$, $(M, \cdot, e) \in \text{Mon}$ and $f \in \text{Set}(\Sigma, M)$, there is a unique monoid homomorphism $\hat{f} \in \text{Mon}(\text{List}(\Sigma), M)$



The theorem just says that $i_\Sigma: \Sigma \rightarrow \text{List}(\Sigma)$ is an initial object in the following category:

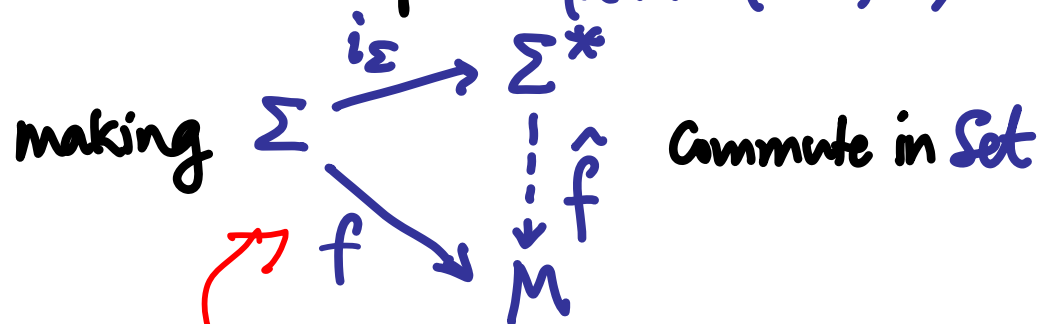
Category Σ/Mon :

- objects (M, f) where $M \in \text{Mon} \& f \in \text{Set}(\Sigma, M)$
- morphisms in $\Sigma/\text{Mon}((M, f), (N, g))$ are $h \in \text{Mon}(M, N)$ s.t.

$$\begin{array}{ccc} \Sigma & \xrightarrow{f} & M \\ & \searrow g & \downarrow h \\ & & N \end{array} \quad \text{Commutates in Set}$$
- identities & composition as in Mon

Example: free monoids as initial objects

Theorem Given $\Sigma \in \text{Set}$, $(M, \cdot, 1) \in \text{Mon}$ and $f \in \text{Set}(\Sigma, M)$, there is a unique monoid homomorphism $\hat{f} \in \text{Mon}(\Sigma^*, M)$



The theorem just says that $i_\Sigma: \Sigma \rightarrow \Sigma^*$ is an initial object in Σ/Mon .

So this universal property determines $\text{List}(\Sigma)$ uniquely up to monoid isomorphism.

We'll see later that $\Sigma \mapsto \text{List}(\Sigma)$ is part of a functor (= category morphism) which is left adjoint to the "forgetful functor" $\text{Mon} \rightarrow \text{Set}$.