Category-theoretic properties

"Any two isomorphic objects in a category have the same category-theoretic properties."

instead of formalizing the "language & logic of category theory", we'll just look at examples of category-theoretic properties.

Here's our first one...

Terminal objects

An object $T \in \mathbb{C}$ of a category $\mathbb{C}$ is **terminal** if for all $X \in \mathbb{C}$, there is a unique morphism $X \to T$ (we'll write $\langle\rangle_X$, or just $\langle\rangle$ for this morphism)

__Theorem__ In a category $\mathbb{C}$:

(a) if T is terminal & $T \cong T'$, then T' is terminal

(b) if T & T' are both terminal, then $T \cong T'$ (and there is only one isomorphism between T & T')

terminal objects are unique up to unique isomorphism

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__Theorem__ In a category $\mathbb{C}$:

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(b) if T & T' are both terminal, then $T \cong T'$
(and ...)

__Proof__ ...

Examples of terminal objects

- In **Set**: any one-element set
- Any one-element set has a unique pre-order & this makes it terminal in **Pre**
- Ditto for **Mon**.
- A pre-ordered set $(P, \leq)$, regarded as a category, has a terminal object iff it has a greatest element: $(\forall x \in P)\ x \leq \top$
- When does a monoid $(M, \cdot, 1)$, regarded as a category, have a terminal object?

The opposite of a category $\mathbb{C}$

is the category $\boxed{\mathbb{C}^{op}}$ defined by

- $\mathrm{Obj}\,\mathbb{C}^{op} \triangleq \mathrm{Obj}\,\mathbb{C}$
- $\mathbb{C}^{op}(X,Y) \triangleq \mathbb{C}(Y,X)$ for all objects X & Y

same objects

same morphisms, but with direction reversed, ~~that~~ is, dom & cod swapped

E.g.

$\mathbb{C}$

$$\begin{tikzcd} 0 \arrow[loop above, "id_0"] \arrow[r, "f"] & 1 \arrow[loop above, "id_1"] \end{tikzcd}$$

$\mathbb{C}^{op}$

$$\begin{tikzcd} 0 \arrow[loop above, "id_0"] & 1 \arrow[loop above, "id_1"] \arrow[l, "f"'] \end{tikzcd}$$

The opposite of a category $\mathbb{C}$

is the category $\boxed{\mathbb{C}^{op}}$ defined by

- $\mathrm{Obj}\,\mathbb{C}^{op} \triangleq \mathrm{Obj}\,\mathbb{C}$
- $\mathbb{C}^{op}(X,Y) \triangleq \mathbb{C}(Y,X)$ for all objects X & Y
- identity morphism on $X \in \mathrm{Obj}\,\mathbb{C}^{op}$ is id_X, the identity on $X \in \mathrm{Obj}\,\mathbb{C}$
- the composition of $f \in \mathbb{C}^{op}(X,Y)$ & $g \in \mathbb{C}^{op}(Y,Z)$ is given by composition $f \circ g \in \mathbb{C}(Z,X)$ in $\mathbb{C}$

$$g \circ_{\mathbb{C}^{op}} f \triangleq f \circ_{\mathbb{C}} g$$

(associativity & unity props hold, because they do in $\mathbb{C}$)

Principle of Duality

Whenever we $\left\{\begin{array}{l}\text{define a concept}\\ \text{prove a theorem}\end{array}\right.$ in terms of commutative diagrams, we obtain another $\left\{\begin{array}{l}\text{concept}\\ \text{theorem}\end{array}\right.$, called its **dual**, by reversing the direction of morphisms throughout (i.e. by replacing $\mathbb{C}$ by $\mathbb{C}^{op}$).

For example...

Initial object

is the dual notion to "terminal object"

An object $I \in \mathbb{C}$ of a category $\mathbb{C}$ is **initial** if for all $X \in \mathbb{C}$, there is a unique morphism $I \to X$ (we'll write $[]_X$, or just $[]$ for this morphism)

By duality, we have ~~that~~ initial objects are unique up to iso and that any object isomorphic to an initial object is itself initial.

NB "isomorphism" is a self-dual concept

Examples of initial objects

- The empty set is initial in **Set**
- Any one-element monoid (has uniquely determined monoid operation & unit) is initial in **Mon** (why?)

(so initial & terminal objects coincide in **Mon**)

an object that's both initial & terminal is sometimes called a <u>zero object</u>

Example: free monoids as initial objects

(relevant to automata & formal languages)

Free monoid on a set $\Sigma \in \mathbf{Set}$:

$$(List(\Sigma), @, nil)$$

empty list

Set of finite lists of elements of Σ

list concatenation:

$nil @ \ell' = \ell'$

$(a::\ell) @ \ell' = a::(\ell @ \ell')$

Example: free monoids as initial objects

Free monoid on a set $\Sigma \in \mathbf{Set}$:

$$i_\Sigma : \Sigma \to List(\Sigma) \text{ in } \mathbf{Set}$$

$$a \mapsto [a] \quad \text{where } [a] \triangleq a :: nil$$

i_Σ sends element $a \in \Sigma$ to corr. list of length 1

It has the following "universal property"...

Example: free monoids as initial objects

Theorem Given $\Sigma \in \mathbf{Set}$, $(M, \bullet, e) \in \mathbf{Mon}$ and $f \in \mathbf{Set}(\Sigma, M)$, there is a unique monoid homomorphism $\hat{f} \in \mathbf{Mon}(\mathrm{List}(\Sigma), M)$

making
$$
\begin{tikzcd}
 & \mathrm{List}(\Sigma) \arrow[d, "\hat{f}"] \\
\Sigma \arrow[ur, "i_\Sigma"] \arrow[r, "f"'] & M
\end{tikzcd}
$$
commute in $\mathbf{Set}$

Proof ...

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Theorem Given $\Sigma \in \mathrm{Set}$, $(M, \cdot, e) \in \mathrm{Mon}$ and $f \in \mathrm{Set}(\Sigma, M)$, there is a unique monoid homomorphism $\hat{f} \in \mathrm{Mon}(\mathrm{List}(\Sigma), M)$ making

$$\begin{tikzcd} \Sigma \arrow[r, "i_\Sigma"] \arrow[dr, "f"'] & \mathrm{List}(\Sigma) \arrow[d, "\hat{f}"] \\ & M \end{tikzcd}$$

commute in Set

The theorem just says that $i_\Sigma : \Sigma \to \mathrm{List}(\Sigma)$ is an initial object in the following category:

Category Σ/Mon :

- objects (M, f) where $M \in \mathrm{Mon}$ & $f \in \mathrm{Set}(\Sigma, M)$
- morphisms in $\Sigma/\mathrm{Mon}((M,f),(N,g))$ are $h \in \mathrm{Mon}(M,N)$ s.t.

$$\begin{tikzcd} & M \arrow[d, "h"] \\ \Sigma \arrow[ur, "f"] \arrow[r, "g"'] & N \end{tikzcd}$$

commutes in Set

- identities & composition as in Mon

Example: free monoids as initial objects

Theorem Given $\Sigma \in \mathbf{Set}$, $(M, \cdot, 1) \in \mathbf{Mon}$ and $f \in \mathbf{Set}(\Sigma, M)$, there is a unique monoid homomorphism $\hat{f} \in \mathbf{Mon}(\Sigma^*, M)$ making

$$\begin{tikzcd} \Sigma \arrow[r, "i_\Sigma"] \arrow[dr, "f"'] & \Sigma^* \arrow[d, dashed, "\hat{f}"] \\ & M \end{tikzcd}$$

commute in $\mathbf{Set}$

The theorem just says that $i_\Sigma : \Sigma \to \Sigma^*$ is an initial object in $\Sigma / \mathbf{Mon}$.

So this universal property determines $\mathrm{List}(\Sigma)$ uniquely up to monoid isomorphism.

We'll see later that $\Sigma \mapsto \mathrm{List}(\Sigma)$ is part of a functor (= category morphism) which is left adjoint to the "forgetful functor" $\mathbf{Mon} \to \mathbf{Set}$.