

Theories as Categories

Recall from earlier:

$$\text{algebraic theory} \xrightarrow{\quad \mathbb{T} \quad} \text{Cartesian category } \mathcal{C}_{\mathbb{T}}$$

The same construction applied to an equational theory in STLC yields a CCC.

(Exponential of $\Gamma = [x_1 : A_1, \dots, x_n : A_n]$ & $\Delta = [y_1 : B_1, \dots, y_m : B_m]$ in $\mathcal{C}_{\mathbb{T}}$ is $[f_1 : \vec{A} \rightarrow B_1, \dots, f_m : \vec{A} \rightarrow B_m]$, where $\vec{A} \rightarrow B \triangleq A_1 \rightarrow (A_2 \rightarrow \dots \rightarrow (A_n \rightarrow B) \dots)$)

Theories as Categories

Recall from earlier:

$$\text{algebraic theory} \xrightarrow{\quad \mathbb{T} \quad} \text{Cartesian category } \mathcal{C}_{\mathbb{T}}$$

The same construction applied to an equational theory in STLC yields a CCC, but there's an equivalent & simpler construction ...

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Theories as Categories

algebraic theory
over STC
 $\frac{\Pi}{\vdash}$ \rightarrow C.C.C
 C_Π

- objects of C_Π are types of Π
- morphisms $A \rightarrow B$ in C_Π are equivalence classes of closed terms
 - $\vdash t : A \rightarrow B$ for equiv. relation

$t \sim t'$ if $\vdash t = t' : A \rightarrow B$ is a Π theorem

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LOGIC

TYPE THEORY

CATEGORY TH.

propositions \leftrightarrow types \leftrightarrow objects

proofs \leftrightarrow terms \leftrightarrow morphisms

E.g. I_PL vs S_TLC vs CCC's
proofs terms morphisms

Recall the derivation of $\varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in
IPL :

$$\frac{\frac{\frac{\frac{\varphi \vdash \psi \Rightarrow \theta}{\varphi \vdash \psi \Rightarrow \theta} (\text{Ax})}{\varphi \vdash \psi} (\text{Ax})}{\varphi \vdash \psi} (\Rightarrow E)}{\varphi \vdash \psi \Rightarrow \theta} (\Rightarrow E)$$
$$\frac{\frac{\frac{\varphi \vdash \psi \Rightarrow \psi}{\varphi \vdash \psi} (\text{Ax})}{\varphi \vdash \psi} (\Rightarrow E)}{\varphi \vdash \psi} (\Rightarrow E)$$
$$\frac{\varphi \vdash \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}{\varphi \vdash \psi} (\Rightarrow I)$$

$$(\overline{\varPhi} \triangleq \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi)$$

A corresponding STLC term:

$$(\Phi \triangleq (y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi])$$

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these correspondences can be made into categorical equivalences - need the notions of functor & natural transformation to define "equivalence" ...

↳ E.g. IPL vs STLC vs CCC's
cut-free proofs terms morphisms

Functors

are the appropriate notion of morphism between categories. Given categories \mathbb{C} & \mathbb{D} , a **functor** $F : \mathbb{C} \rightarrow \mathbb{D}$ is specified by :

- a function $\text{Obj } \mathbb{C} \rightarrow \text{Obj } \mathbb{D}$ whose value at a \mathbb{C} -object X is written Fx
- for all \mathbb{C} -objects X & Y , a function $\mathbb{C}(X, Y) \rightarrow \mathbb{D}(Fx, Fy)$ whose value at a \mathbb{C} -morphism $f : X \rightarrow Y$ is written $Ff : Fx \rightarrow Fy$

satisfying { $F(g \circ f) = Fg \circ Ff$
 $F(id_X) = id_{Fx}$

Examples of functors

"forgetful" functors from categories of sets-with-structure back to Set

E.g. $U : \text{Mon} \rightarrow \text{Set}$

$$\begin{cases} U(M, \circ, e) \stackrel{\Delta}{=} M \\ U((M, \circ, e) \xrightarrow{f} (N, \circ, e)) \stackrel{\Delta}{=} M \xrightarrow{f} N \end{cases}$$

and similarly $U : \text{Pre} \rightarrow \text{Set}$

Examples of functors

Free monoid functor

$$F : \text{Set} \rightarrow \text{Mon}$$

Recall free monoid on a set Σ is

$$(\text{List}(\Sigma), @, \text{nil})$$

finite lists of
elements of Σ

list concatenation

empty list

Examples of functors

Free monoid functor

$$F : \text{Set} \rightarrow \text{Mon}$$

Recall free monoid on a set Σ is

$$F(\Sigma) \triangleq (\text{List}(\Sigma), \circ, \text{nil})$$

Given $f \in \text{Set}(\Sigma_1, \Sigma_2)$ we get

$F(f) : F(\Sigma_1) \rightarrow F(\Sigma_2)$ mapping each list

$$l = [a_1, \dots, a_n] \in \Sigma_1^* \text{ to } Ff l \triangleq [f a_1, \dots, f a_n]$$

Easy to see that $F(\text{id}_{\Sigma}) = \text{id}_{F(\Sigma)}$ &

$$F(g \circ f) = (Fg) \circ (Ff)$$

Examples of functors

If \mathbb{C} is a cartesian category and $X \in \text{Obj } \mathbb{C}$, then

$$Y \in \text{Obj } \mathbb{C} \mapsto Y \times X$$

extends to a functor

$$(-) \times X : \mathbb{C} \rightarrow \mathbb{C}$$

Via $(Y \xrightarrow{f} Y') \mapsto (Y \times X \xrightarrow{fx \circ id_X} Y' \times X)$

since $\begin{cases} id_Y \times id_X = id_{Y \times X} \\ (g \circ f) \times id_X = (g \times id_X) \circ (f \times id_X) \end{cases}$

[See Ex-Sh. 2, q1c]

Examples of functors

If \mathbb{C} is a cartesian closed category and $X \in \text{Obj } \mathbb{C}$, then

$$Y \in \text{Obj } \mathbb{C} \mapsto Y^X$$

extends to a functor

$$(-)^X : \mathbb{C} \rightarrow \mathbb{C}$$

Via $(Y \xrightarrow{f} Y') \mapsto (Y^X \xrightarrow{f^X} Y'^X)$

Since $\begin{cases} id^X = id \\ (g \circ f)^X = g^X \circ f^X \end{cases}$ "cur(f o app)"

[See Ex. Sh. 3, q 7]

Contravariance

A functor $F : \mathbb{C}^{\text{op}} \rightarrow \mathbb{D}$ is called a
contravariant functor from \mathbb{C} to \mathbb{D}

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathbb{C}

then $X \xleftarrow{Ff} Y \xleftarrow{Fg} Z$ in \mathbb{C}^{op}

so

$Fx \xleftarrow{Ff} FY \xleftarrow{Fg} FZ$ in \mathbb{D}



$$F(g \circ f) = Ff \circ Fg$$

composition in \mathbb{C}



composition
in \mathbb{D}

Example of contravariant functor

If \mathbb{C} is a cartesian closed category and $X \in \text{Obj } \mathbb{C}$, then

$$Y \in \text{Obj } \mathbb{C} \mapsto X^Y$$

extends to a functor

$$X^{(-)} : \mathbb{C}^{\text{op}} \rightarrow \mathbb{C}$$

Via $(Y \xrightarrow{f} Y') \mapsto (X^Y \xleftarrow[X^f]{\parallel} X^{Y'})$

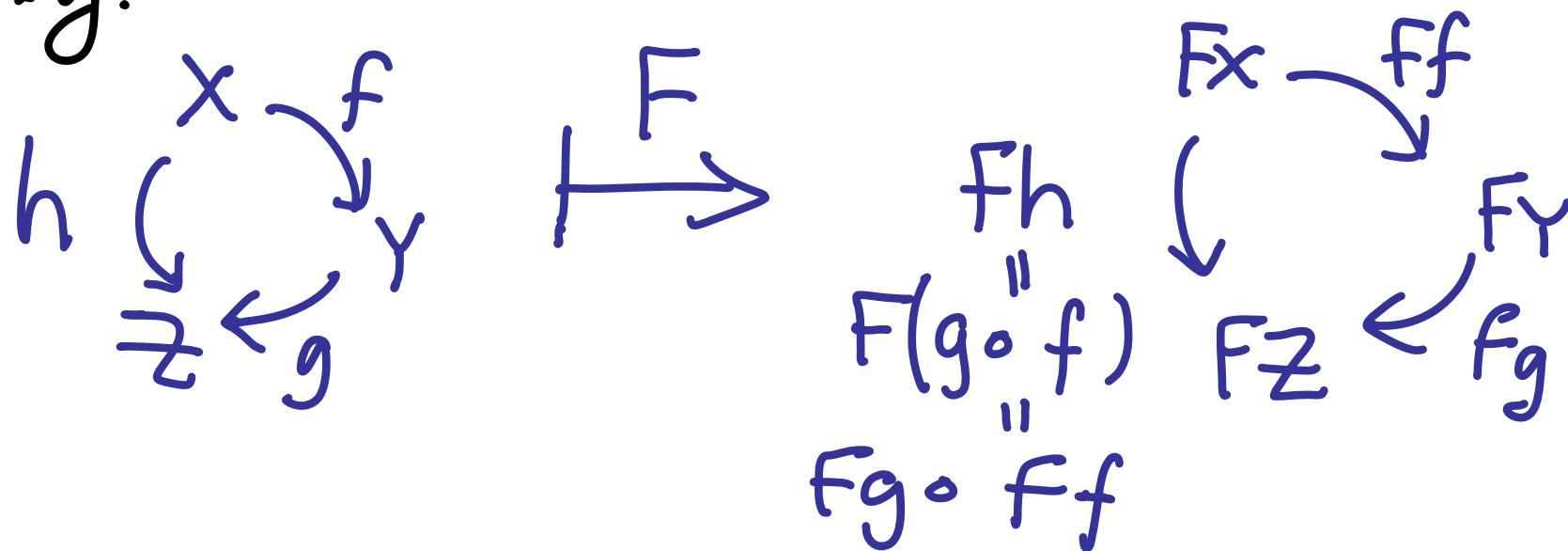
Since $\begin{cases} X^{\text{id}} = \text{id} \\ X^{g \circ f} = X^f \circ X^g \end{cases}$ Cwr(appo(idxf))

[See Ex. Sh. 3, qd]

Note that since a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ preserves domains, codomains, composition

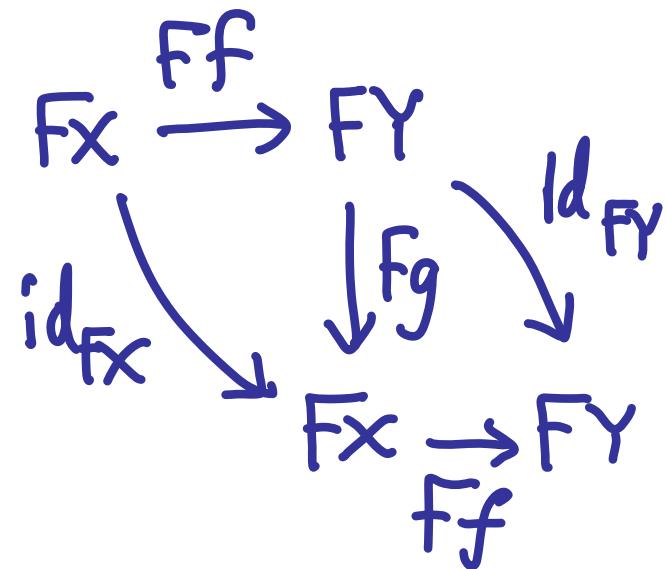
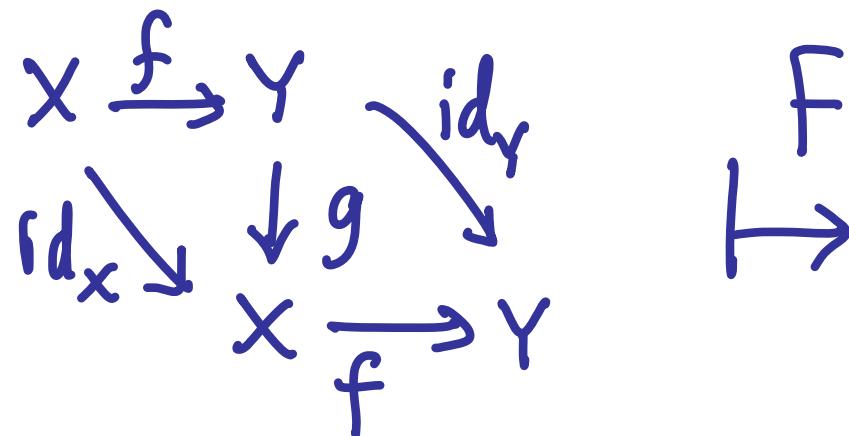
it sends commutative diagrams in \mathcal{C} to commutative diagrams in \mathcal{D}

E.g.



Note that since a functor $F: \mathbb{C} \rightarrow \mathbb{D}$ preserves domains, codomains, composition & identities,

it sends isomorphisms in \mathbb{C} to isos in \mathbb{D} because



So
$$F(f^{-1}) = (Ff)^{-1}$$

Composing functors

Given functors

$$\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$$

we get a functor

$$G \circ F : \mathbb{C} \rightarrow \mathbb{E}$$

$$G \circ F \left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \right) \stackrel{\Delta}{=} \begin{array}{c} G(FX) \\ \downarrow \\ G(Ff) \\ G(FY) \end{array}$$

(This preserves composition & identities because
 F & G do so)

Identity functor

on a category \mathbb{C} is

$$\boxed{\text{Id}_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}}$$

where

$$\text{Id}_{\mathbb{C}} \left(\begin{smallmatrix} X \\ \downarrow f \\ Y \end{smallmatrix} \right) \stackrel{\Delta}{=} \begin{smallmatrix} X \\ \downarrow f \\ Y \end{smallmatrix}$$

Functor composition satisfies the usual
Category laws

associativity $H \circ (G \circ F) = (H \circ G) \circ F$

unity $Id_D \circ F = F = F \circ Id_C$

So we can get categories whose
objects are categories
morphisms are functors
but we have to be a bit careful about size...