1. [8/40 marks] Let $V$ be the category with three distinct objects $L, P, R$ and whose only non-identity morphisms are $p : P \to L$ and $q : P \to R$.

(a) Complete the definition of $V$ by giving the nine sets $V(X,Y)$ of morphisms between pairs of objects $X,Y \in \{L, P, R\}$ and defining the composition operations.

(b) Do either of $V$ or $V^{op}$ have a terminal object?

(c) Do either of $V$ or $V^{op}$ have binary products? [Hint: recall that in a pre-ordered set regarded as a category, products are given by greatest lower bounds.]

2. [8/40 marks] Let $\Sigma = \{a, b\}$ be a two-element set ($a \neq b$) and let $\_ \oplus \_ : \Sigma \times \Sigma \to \Sigma$ and $\_ \otimes \_ : \Sigma \times \Sigma \to \Sigma$ be binary operations on $\Sigma$ defined by the following tables:

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<th>$a$</th>
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(so that $a \oplus b = a$, $a \otimes b = b$, etc.)

(a) Show that for suitable choices of elements $1_M, 1_N \in \Sigma$, there are monoids $M = (\Sigma, \oplus, 1_M)$ and $N = (\Sigma, \otimes, 1_N)$.

(b) Show that $M$ and $N$ are not isomorphic in the category $\text{Mon}$ of monoids and monoid homomorphisms. [Hint: assume they are isomorphic and derive a contradiction.]

3. [6/40 marks] Let $C$ be a category with binary products. Given a $C$-object $X$, the diagonal morphism $\delta_X \in C(X, X \times X)$ and the twist morphism $\tau_X \in C(X \times X, X \times X)$ are defined by:

$$\delta_X \triangleq (\text{id}_X, \text{id}_X)$$

$$\tau_X \triangleq (\pi_2, \pi_1)$$

(a) For each $f \in C(X,Y)$, show that $\delta_Y \circ f = (f \times f) \circ \delta_X \in C(X,Y \times Y)$ (where $f \times f$ denotes the product of morphisms introduced in Ex. Sh. 2, question 1b).

(b) Show that $\tau_X \circ \delta_X = \delta_X$.

(c) Show that $\tau_X \circ \tau_X = \text{id}_{X \times X}$.

4. [4/40 marks] Consider an algebraic signature with

- two sorts $D$ and $L$
- two function symbols $\text{emp} : [] \to L$ and $\text{cons} : [D, L] \to L$

A particular structure for this signature in a cartesian category $C$ interprets $D$ and $L$ as the $C$-objects $X$ and $Y$ respectively; and interprets the function symbols $\text{emp}$ and $\text{cons}$ as the morphisms $e \in C(1, Y)$ and $c \in C(X \times Y, Y)$ respectively (where 1 is terminal in $C$). With respect to this structure, give the morphisms in $C$ that are the interpretation of the following valid typing judgements:
(a) \( x : D, y : L \vdash \text{cons}(x, \text{cons}(x, y)) : L \)
(b) \( x : D, y : L \vdash \text{cons}(x, \text{cons}(x, \text{emp}())) : L \)

5. [8/40 marks] Let \( C \) be a category. A pair of \( C \)-morphisms is said to be parallel if their domains are equal and their codomains are equal. Given such a parallel pair of morphisms \( f, g \in C(X, Y) \), an equalizer for \( f \) and \( g \) is by definition a \( C \)-morphism \( e : E \to X \) satisfying \( f \circ e = g \circ e \) and with the following property:

For all \( C \)-objects \( Z \) and morphisms \( h \in C(Z, X) \), if \( f \circ h = g \circ h \in C(Z, Y) \), then there exists a unique morphism \( k \in C(Z, E) \) satisfying \( e \circ k = h \).

(a) Show that every equalizer is a monomorphism (see Ex. Sh. 1, question 4).
(b) Suppose that \( f \in C(X, Y) \) is a split monomorphism, that is, there is a morphism \( g \in C(Y, X) \) with \( g \circ f = \text{id}_X \) (see Ex. Sh. 1, question 4). Show that \( f : X \to Y \) is the equalizer of the parallel pair \( f \circ g \) and \( \text{id}_Y \).
(c) Show that the category \( \text{Set} \) of sets and functions possesses equalizers for all parallel pairs of morphisms.

6. [6/40 marks] Let \( X \) be an object of a category \( C \). The slice category \( C/X \) is defined by:

- The objects of \( C/X \) are pairs \( (A, p) \) where \( A \in \text{obj} \ C \) and \( p \in C(A, X) \).
- Given two such objects \( (A, p) \) and \( (B, q) \), a morphism \( f : (A, p) \to (B, q) \) in \( C/X \) is a \( C \)-morphism \( f : A \to B \) such that \( q \circ f = p \)

- Composition and identities in \( C/X \) are given by those in \( C \).

(a) Show that \( C/X \) always has a terminal object.
(b) When \( C = \text{Set} \), the category of sets and functions, show that \( \text{Set}/X \) has binary products. [Hint: given \( (A, p), (B, q) \in \text{obj}(\text{Set}/X) \), consider a suitable subset of \( \{(a, b) \mid a \in A \land b \in B\} \).]