1. Show that the category \( \text{Pre} \) of pre-ordered sets and monotone functions is a cartesian closed category.

2. Using the natural deduction rules for Intuitionistic Propositional Logic (given in Lecture 8), give proofs of the following judgements. In each case write down a corresponding typing judgement of the Simply Typed Lambda Calculus.

   (a) \( \psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi \)
   
   (b) \( \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi \)
   
   (c) \((\varphi \Rightarrow \psi) \Rightarrow \psi \Rightarrow \varphi \Rightarrow \psi \)

3. (a) Give terms \( s \) and \( t \) of the Simply Typed Lambda Calculus that satisfy the following typing and \( \beta\eta \)-equality judgements:

   \[
   x : (A \times B) \rightarrow C \vdash s : A \rightarrow (B \rightarrow C) \tag{1}
   \]
   
   \[
   y : A \rightarrow (B \rightarrow C) \vdash t : (A \times B) \rightarrow C \tag{2}
   \]
   
   \[
   x : (A \times B) \rightarrow C \vdash t[s/y] =_{\beta\eta} x : (A \times B) \rightarrow C \tag{3}
   \]
   
   \[
   y : A \rightarrow (B \rightarrow C) \vdash s[t/x] =_{\beta\eta} y : A \rightarrow (B \rightarrow C) \tag{4}
   \]

   (b) Explain why question (3a) implies that for any three objects \( X,Y \) and \( Z \) in a cartesian closed category \( \mathcal{C} \), there are morphisms

   \[
   f : Z^{(X \times Y)} \rightarrow (Z^Y)^X \tag{5}
   \]
   
   \[
   g : (Z^Y)^X \rightarrow Z^{(X \times Y)} \tag{6}
   \]

   that give an isomorphism \( Z^{(X \times Y)} \cong (Z^Y)^X \) in \( \mathcal{C} \).

4. Make up and solve a question like question 3 ending with an isomorphism \( X^1 \cong X \) in any cartesian closed category \( \mathcal{C} \).

5. Suppose that \( \text{app} : Y^X \times X \rightarrow Y \) is an exponential for objects \( X \) and \( Y \) in a cartesian category \( \mathcal{C} \). Show that \( \text{cur}(\text{app}) = \text{id}_{Y^X} \). [Hint: recall from equation (4) on Exercise Sheet 2 that \( \text{id}_{Y^X} \times \text{id}_X = \text{id}_{Y^X \times X} \).]

6. Suppose \( f : Y \times X \rightarrow Z \) and \( g : W \rightarrow Y \) are morphisms a cartesian closed category \( \mathcal{C} \). Prove that

   \[
   \text{cur}(f \circ (g \times \text{id}_X)) = (\text{cur} f) \circ g \in \mathcal{C}(W, Z^X) \tag{7}
   \]

   [Hint: use Ex.Sh. 2, question 1c.]

7. Let \( \mathcal{C} \) be a cartesian closed category. For each \( \mathcal{C} \)-object \( X \) and \( \mathcal{C} \)-morphism \( f : Y \rightarrow Z \), define

   \[
   f^X \triangleq \text{cur}(Y^X \times X \overset{\text{app}}{\rightarrow} Y \overset{f}{\rightarrow} Z) \in \mathcal{C}(Y^X, Z^X) \tag{8}
   \]

   (a) Prove that \( (\text{id}_Y)^X = \text{id}_{Y^X} \).
(b) Given $f \in C(Y \times X, Z)$ and $g \in C(Z, W)$, prove that
$$\text{cur}(g \circ f) = g^X \circ \text{cur} f \in C(Y, W^X) \quad (9)$$

(c) Deduce that if $u \in C(Y, Z)$ and $v \in C(Z, W)$, then $(v \circ u)^X = v^X \circ u^X \in C(Y^X, W^X)$.

[Hint: for part (7a) use question 5; for part (7b) use Ex.Sh. 2, question 1c.]

8. Let $C$ be a cartesian closed category. For each $C$-object $X$ and $C$-morphism $f : Y \to Z$, define
$$X^f \triangleq \text{cur}(X^Z \times Y \xrightarrow{id \times f} X^Z \times Z \xrightarrow{\text{app}} X) \in C(X^Z, X^Y) \quad (10)$$

(a) Prove that $X^{id_Y} = id_{X^Y}$.

(b) Given $g \in C(W, X)$ and $f \in C(Y \times X, Z)$, prove that
$$\text{cur}(f \circ (id_Y \times g)) = Z^X \circ \text{cur} f \in C(Y, Z^W) \quad (11)$$

(c) Deduce that if $u \in C(Y, Z)$ and $v \in C(Z, W)$, then $X^{(v \circ u)} = X^u \circ X^v \in C(X^W, X^Y)$.

[Hint: for part (8a) use question 5; for part (8b) use Ex.Sh. 2, question 1c.]

9. Let $C$ be a cartesian closed category in which every pair of objects $X$ and $Y$ possesses a binary coproduct $X \xrightarrow{\text{inl}_X} X + Y \xleftarrow{\text{inr}_X} Y$. For all objects $X, Y, Z \in C$ construct an isomorphism $(Y + Z) \times X \cong (Y \times X) + (Z \times X)$. [Hint: you may find it helpful to use some of the properties from question 7.]