$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "\exists g"] \end{tikzcd}$$

$g \circ f = id_A \iff \forall a \in A.\ g(fa) = a$

$f \circ g = id_B \iff \forall b \in B.\ f(gb) = b$

Definition 130 *Two sets* A *and* B *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$\{0\} \not\cong \{0,1\}$

$$\mathrm{A} \cong \mathrm{B} \quad \textit{or} \quad \#\mathrm{A} = \#\mathrm{B} \quad .$$

NB. If f has an inverse Then it is unique and typically denoted f^{-1}

NB: $A \cong A$, $A \cong B \Rightarrow B \cong A$, $A \cong B, B \cong C \Rightarrow A \cong C$

Examples:

1. $\{0,1\} \cong \{\mathbf{false}, \mathbf{true}\}$.

2. $\mathbb{N} \cong \mathbb{N}^{+}$, $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \cong \mathbb{Q}$.

$(f' \circ f)^{-1} = f^{-1} \circ (f')^{-1}$

$\mathbb{N} \times \mathbb{N}$

	0	1	2	...	n	...
0	0	2	3	9	10	
1	1	4	8	11		
2	5	7	12			
⋮	6	13				
m	14					
⋮	15					

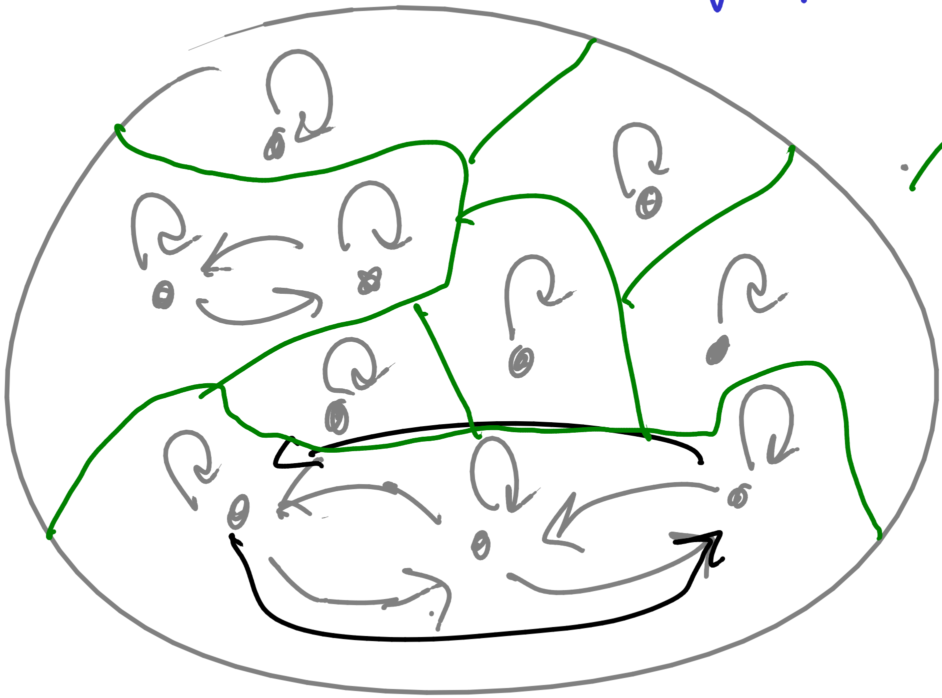
(m,n)

Equivalence relations and set partitions

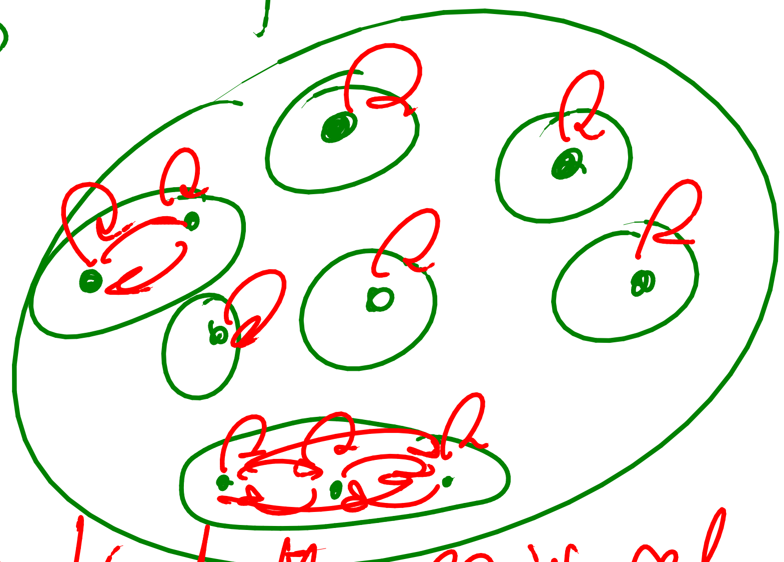
- Equivalence relations.

$R \subseteq A \times A$:
$\forall a \in A.\ aRa$
$\forall a,b \in A.\ aRb \Rightarrow bRa$
$\forall a,b,c \in A.\ aRb \wedge bRc \Rightarrow aRc$

A



⇝ partition of A



reconstruct the equiv. rel

- Set partitions.

$$\underline{\text{Part}}(A) = \{\pi \mid \pi \text{ is a partition}\}$$

$$\pi \subseteq \mathcal{P}(A)$$

$$\pi = \{P_1, P_2, \dots P_i \dots\}$$

$$P_i \subseteq A$$

$$\bigcup_i P_i = A$$

$$\begin{matrix} a \in P_i \\ a \in P_j \end{matrix} \Rightarrow i = j$$

$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A)$$

$R \subseteq A \times A$ equiv. rel $\longmapsto$ $\{ \ldots, B \subseteq A \ldots \} \stackrel{\mathrm{def}}{=} \pi(R)$

need define (i.e. prove) That this is a partition.

$$B \in \pi(R) \iff \exists\, a \in A .\ B = [a]_R$$

$$[a]_R = \{ x \mid a R x \}$$

equiv. class of a

$$\underline{\mathrm{EquivRel}}(A) \longleftarrow \underline{\mathrm{Part}}(A)$$

$$R(\pi) \subseteq A \times A \longleftarrow\!\!\shortmid \quad \pi = \{ \dots B \dots \}$$

need to prove our definition is an equiv relation.

$$x \; R(\pi) \; y \iff \exists B \in \pi. \; x \in B \wedge y \in B.$$

<u>Exercise</u>: These two processes are inverses of each other.

Theorem 133 *For every set A,*

$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A) \quad .$$

PROOF:

If f and g are invertible then so is h defined: $h(a,b) = (fa, gb) \in X \times Y$

$a \in A,\ b \in B$

Calculus of bijections

- $A \cong A$, $A \cong B \implies B \cong A$, $(A \cong B \wedge B \cong C) \implies A \cong C$
- If $A \overset{f}{\cong} X$ and $B \overset{g}{\cong} Y$ then

$$\mathcal{P}(A) \cong \mathcal{P}(X) \quad , \quad A \times B \cong X \times Y \quad , \quad A \uplus B \cong X \uplus Y \quad ,$$

$$\mathrm{Rel}(A, B) \cong \mathrm{Rel}(X, Y) \quad , \quad (A \rightharpoonup B) \cong (X \rightharpoonup Y) \quad ,$$

$$(A \Rightarrow B) \cong (X \Rightarrow Y) \quad , \quad \mathrm{Bij}(A, B) \cong \mathrm{Bij}(X, Y)$$

$C : (\alpha * \beta \to \gamma) \to (\alpha \to \beta \to \gamma)$

$C\, f \overset{\text{def}}{=} \text{fn } a \Rightarrow \text{fn } b \Rightarrow f(a,b)$

- $A \cong [1] \times A \;,\; (A \times B) \times C \cong A \times (B \times C) \;,\; A \times B \cong B \times A$
- $[0] \uplus A \cong A \;,\; (A \uplus B) \uplus C \cong A \uplus (B \uplus C) \;,\; A \uplus B \cong B \uplus A$
- $[0] \times A \cong [0] \;,\; (A \uplus B) \times C \cong (A \times C) \uplus (B \times C)$
- $(A \Rightarrow [1]) \cong [1] \;,\; (A \Rightarrow (B \times C)) \cong (A \Rightarrow B) \times (A \Rightarrow C)$
- $([0] \Rightarrow A) \cong [1] \;,\; ((A \uplus B) \Rightarrow C) \cong (A \Rightarrow C) \times (B \Rightarrow C)$
- $([1] \Rightarrow A) \cong A \;,\; \underline{((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))}$
- $(A \rightharpoonup B) \cong (A \Rightarrow (B \uplus [1]))$
- $\boxed{\mathcal{P}(A) \cong (A \Rightarrow [2])}$

$[2] = \{0, 1\}$

$c^{a \cdot b} = (c^b)^a$

Characteristic (or indicator) functions

$$\mathcal{P}(\mathbf{A}) \cong \big(\mathbf{A} \Rightarrow [\mathbf{2}]\big)$$

$$S \subseteq A \longmapsto \chi_S \quad \underline{\mathrm{def}} \quad \chi_S(a) = \begin{cases} 1 & a \in S \\ 0 & a \notin S \end{cases}$$

$$\{ a \in A \mid f(a) = 1 \} \longleftarrow\!\shortmid \; f : A \to \{0,1\}$$

<u>Ex:</u> These two processes are inverses of each other.

Finite cardinality

Definition 135 *A set A is said to be* finite *whenever $A \cong [n]$ for some $n \in \mathbb{N}$, in which case we write $\#A = n$.*

Theorem 136 *For all* $m, n \in \mathbb{N}$,

1. $\mathcal{P}\big([n]\big) \cong [2^n]$
2. $[m] \times [n] \cong [m \cdot n]$
3. $[m] \uplus [n] \cong [m + n]$
4. $\big([m] \rightharpoonup [n]\big) \cong \big[(n+1)^m\big]$
5. $\big([m] \Rightarrow [n]\big) \cong [n^m]$
6. $\mathrm{Bij}\big([n], [n]\big) \cong [n!]$

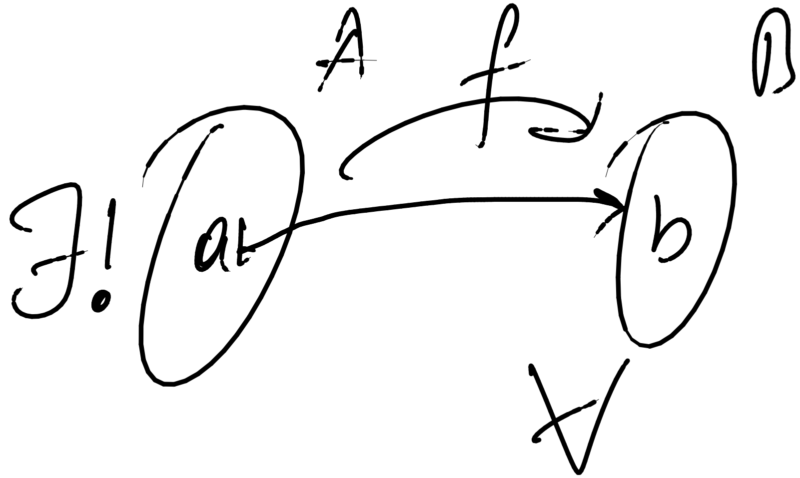
Infinity axiom

$\boxed{\text{There is an infinite set, containing } \emptyset \text{ and closed under successor.}}$

Bijection $\equiv$ Injection $\wedge$ Surjection.

Bijections

Proposition 137 *For a function* $f : A \to B$, *the following are equivalent.*



1. f *is bijective.*

2. $\forall\, b \in B.\, \exists!\, a \in A.\, f(a) = b$.

3. $$\big(\forall\, b \in B.\, \exists\, a \in A.\, f(a) = b\big)$$ $$\wedge$$ $$\big(\forall\, a_1, a_2 \in A.\, f(a_1) = f(a_2) \implies a_1 = a_2\big)$$

Surjection

Injection

Surjections

Definition 138 *A function* $f : A \longrightarrow B$ *is said to be* surjective*, or a* surjection*, and indicated* $f : A \twoheadrightarrow B$ *whenever*

$$\forall\, b \in B.\, \exists\, a \in A.\, f(a) = b \quad .$$

Enumerability

$e(0), e(1), e(2) \cdots \quad e(n) \cdots$

Definition 141

1. *A set* A *is said to be* enumerable *whenever there exists a surjection* $\mathbb{N} \xrightarrow[e]{} \!\!\!\!\!\rightarrow A$*, referred to as an* enumeration*.*
2. *A* countable *set is one that is either empty or enumerable.*

Examples:

1. A bijective enumeration of $\mathbb{Z}$.

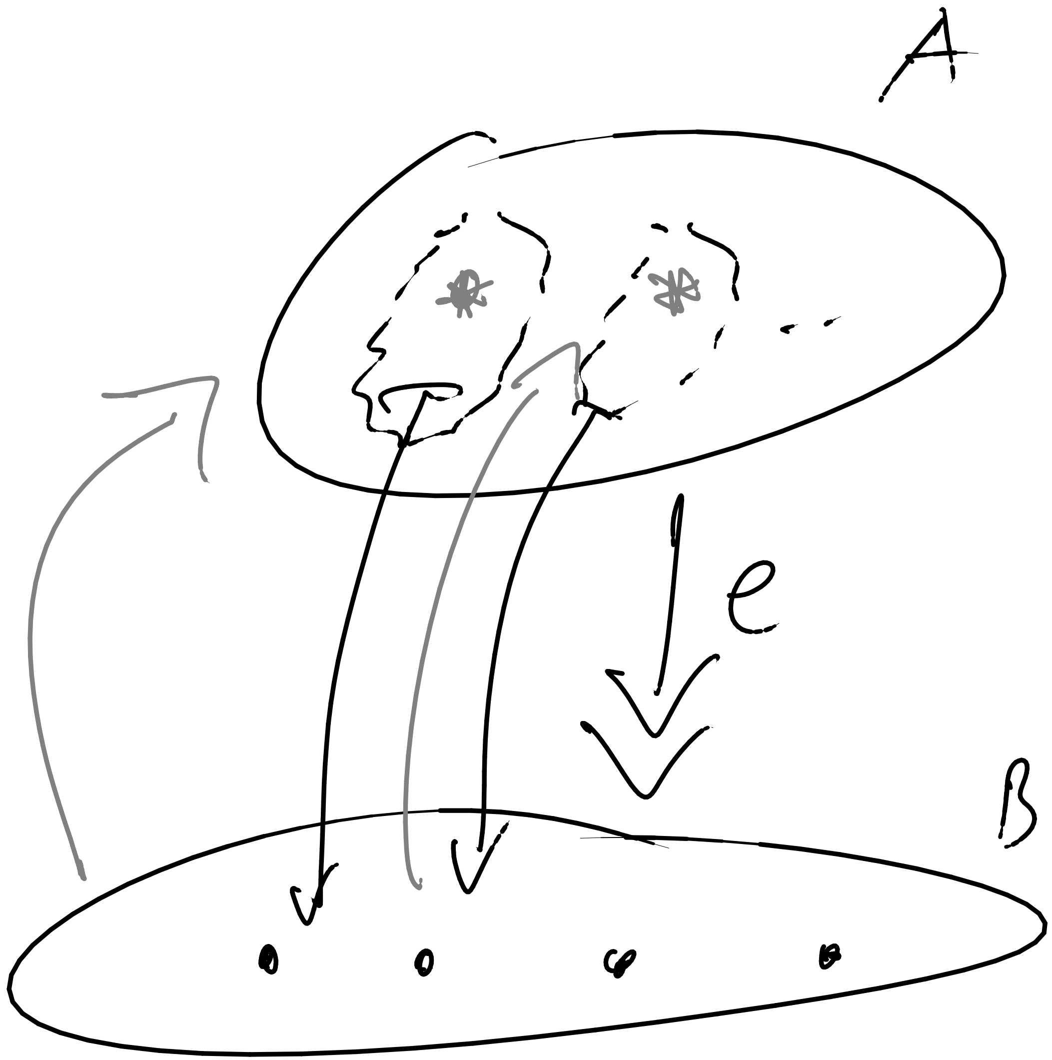
$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$\ldots$	6	4	2	0	1	3	5	$\ldots$

2. A bijective enumeration of $\mathbb{N} \times \mathbb{N}$.

	0	1	2	3	4	5	$\cdots$
0	0	1	5	6			
1	2	4	7				
2	3	8					
3	9						
4							
$\vdots$							

Axiom of choice

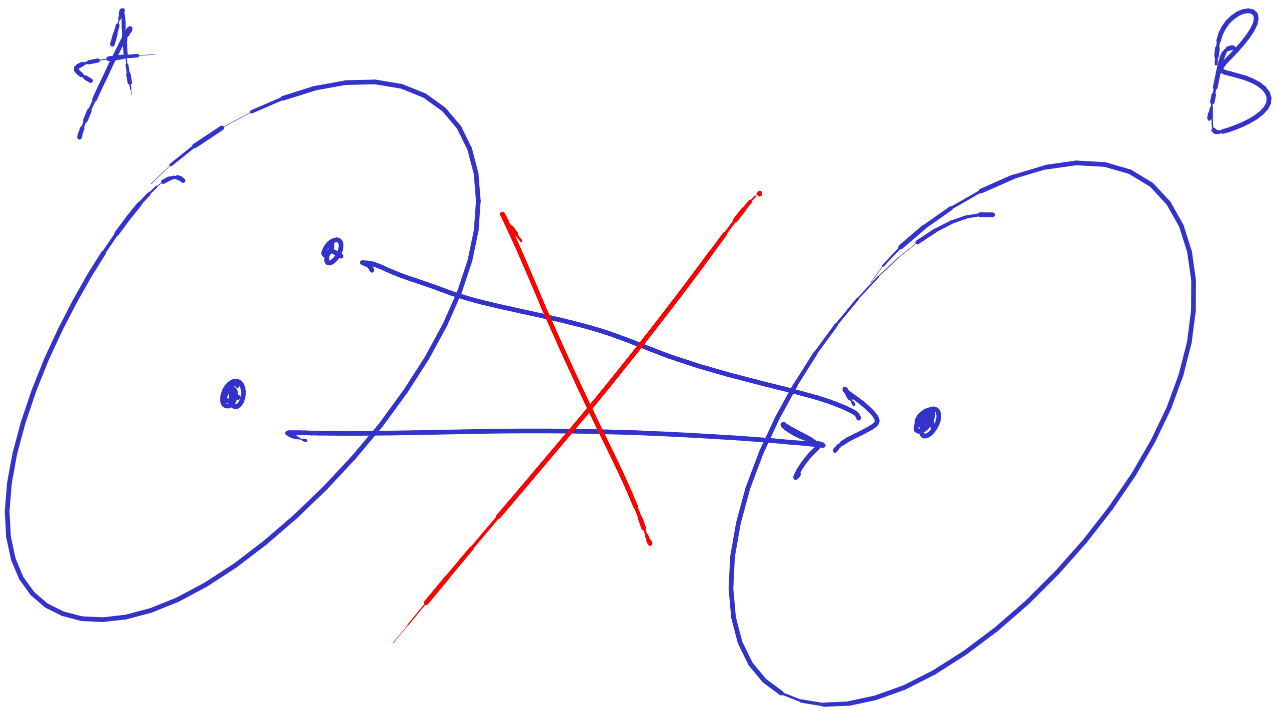
$\boxed{\text{Every surjection has a section.}}$



Injections

Definition 144 *A function* $f : A \to B$ *is said to be* injective, *or an* injection, *and indicated* $f : A \rightarrowtail B$ *whenever*

$$\forall a_1, a_2 \in A.\, \big(f(a_1) = f(a_2)\big) \implies a_1 = a_2 \quad .$$



Replacement axiom

The direct image of every definable functional property on a set is a set.

Set-indexed constructions

For every mapping associating a set A_i to each element of a set I, we have the set

$$\bigcup_{i \in I} A_i \;=\; \bigcup \{A_i \mid i \in I\} \;=\; \{a \mid \exists i \in I.\, a \in A_i\} \quad .$$

Examples:

1. Indexed disjoint unions:

$$\biguplus_{i \in I} A_i \;=\; \bigcup_{i \in I} \{i\} \times A_i$$

2. Finite sequences on a set A:

$$A^* \;=\; \biguplus_{n \in \mathbb{N}} A^n$$

Foundation axiom

$\boxed{\text{The membership relation is well-founded.}}$

Thereby, providing a

Principle of $\in$*-Induction* .