Sets

Intuitively — sets are unordered collection of elements

sets are completely determined by their elements

$A = B \iff \left(\forall x. \; x \in A \iff x \in B \right)$

$A \subseteq B \iff \left(\forall x. \; x \in A \implies x \in B \right)$

We mainly define sets by

$$\{ x \in A \mid P(x) \}$$

$$\emptyset = \{ x \in A \mid \text{false} \}$$

Powerset axiom

For any set, there is a set consisting of all its subsets.

$\mathcal{P}(U)$ ⇝ a set of sets.

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U \quad .$$

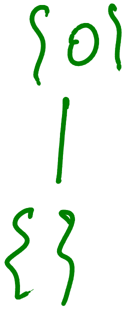
Hasse diagrams

NB. $\emptyset \subseteq A$

$A \subseteq A$

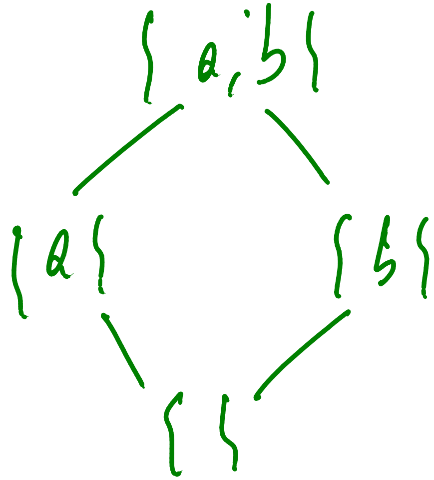
$\mathcal{P}(\{\}) = \{\{\}\}$ $\#\mathcal{P}(\{\}) = 1$

$\mathcal{P}(\{0\}) = \{\{\}, \{0\}\}$ $\#\mathcal{P}(\{0\}) = 2$

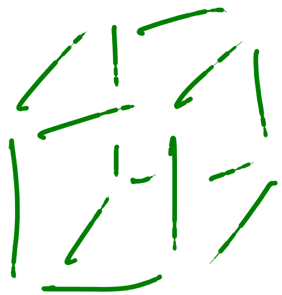


$\mathcal{P}(\{a,b\}) = \{\{\}, \{a,b\}, \{a\}, \{b\}\}$

$\#\mathcal{P}(\{a,b\}) = 4$



$\#\mathcal{P}(\{x,y,z\}) = 8$



Proposition 83 *For all finite sets* U*,*

$$\#\,\mathcal{P}(U) = 2^{\#U} \quad .$$

PROOF IDEA:

Count the number of subsets of a set $U = \{a_1, a_2, \ldots, a_n\}$

$\#\,\mathcal{P}(U) = 2^n$

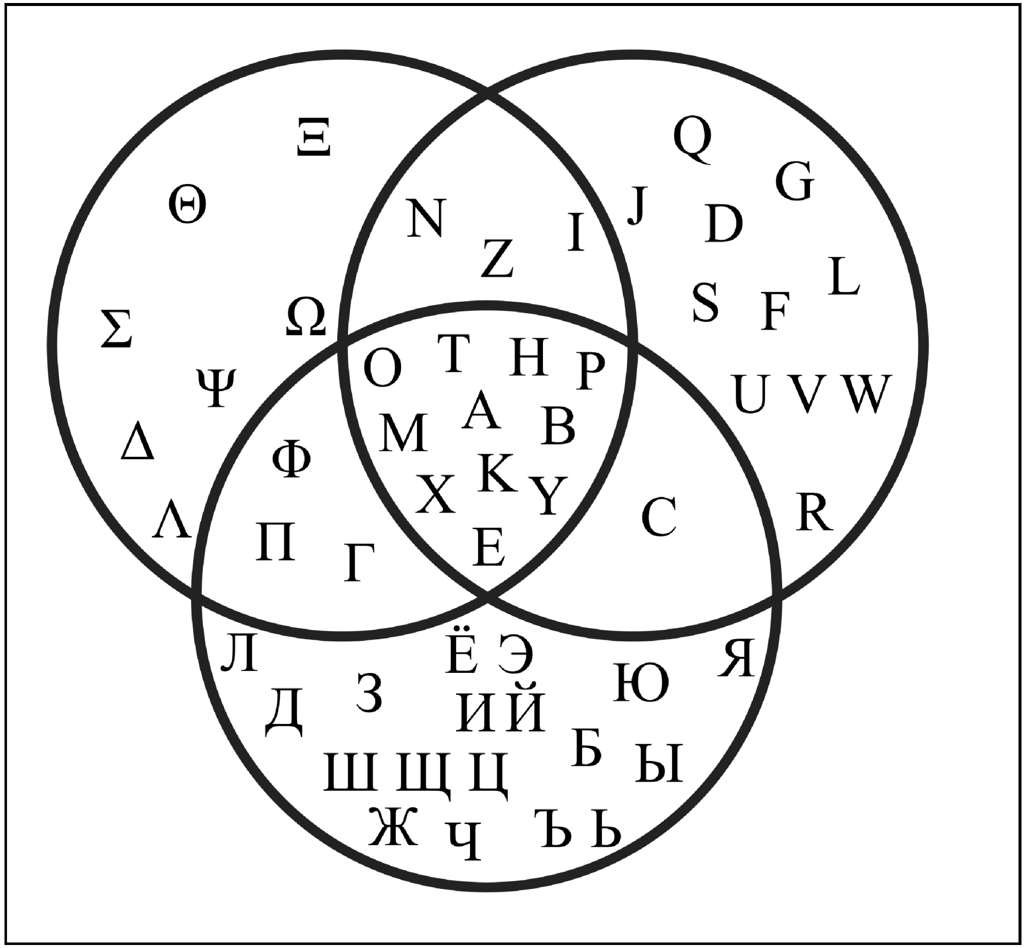
$a_1, a_2, a_3, \ldots, a_n$

$S \subseteq U$ e.g. $S = \{a_1, a_3\} \longmapsto 1\ 0\ 1 \ldots 0 \ldots 0$

$\{a_2, a_4, \ldots, a_{2i}, \ldots\} \longleftarrow 0\,1\,0\,1\,0\,1 \ldots 01$

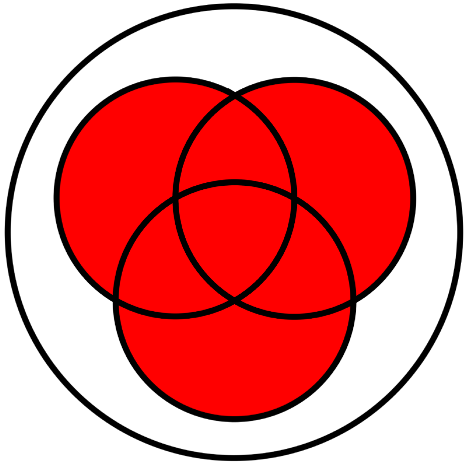
There are 2^n binary seq of length n

Venn diagrams[a]

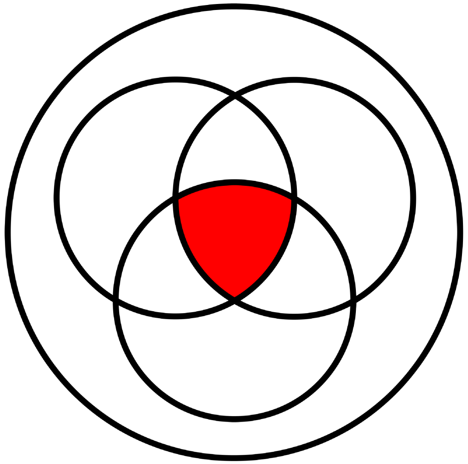


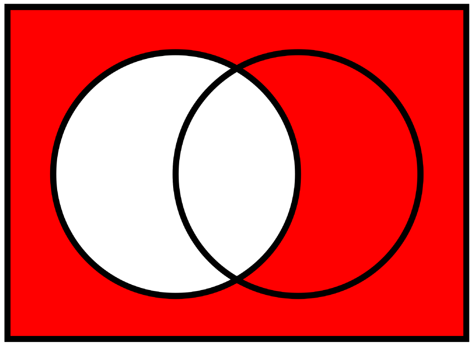
[a]From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

The powerset Boolean algebra

$$\left(\quad \mathcal{P}(U) \;,\quad \emptyset \;,\quad U \;,\quad \cup \;,\quad \cap \;,\quad (\cdot)^c \quad \right)$$

For all $A, B \in \mathcal{P}(U)$,

$$A \cup B = \{ x \in U \mid x \in A \lor x \in B \} \in \mathcal{P}(U)$$

$$A \cap B = \{ x \in U \mid x \in A \land x \in B \} \in \mathcal{P}(U)$$

$$A^c = \{ x \in U \mid \neg (x \in A) \} \in \mathcal{P}(U)$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathcal{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A = A = \mathcal{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathbb{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathbb{U} \cup A = \mathbb{U}$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad , \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) = A = A \cap (A \cup B)$$

- The complement operation $(\cdot)^{\mathrm{c}}$ satisfies complementation laws.

$$A \cup A^{\mathrm{c}} = \mathfrak{U} \quad , \quad A \cap A^{\mathrm{c}} = \emptyset$$

Proposition 84 *Let* U *be a set and let* $A, B \in \mathcal{P}(U)$.

1. $\forall X \in \mathcal{P}(U).\ A \cup B \subseteq X \iff (A \subseteq X \wedge B \subseteq X)$.

2. $\forall X \in \mathcal{P}(U).\ X \subseteq A \cap B \iff (X \subseteq A \wedge X \subseteq B)$.

PROOF:

(1) $X \subseteq U$

($\Rightarrow$) $A \cup B \subseteq X$

RTP: $A \subseteq X$ $\quad B \subseteq X$

Can be shown by observing: $A \subseteq A \cup B$, $B \subseteq A \cup B$

$(X \subseteq Y \wedge Y \subseteq Z) \Rightarrow X \subseteq Z$

($\Leftarrow$) $A \subseteq X$, $B \subseteq X$

RTP: $A \cup B \subseteq X$

RTP: $A \cup B \subseteq X$

$\iff (\forall x.\ x \in A \cup B \Rightarrow x \in X.)$

Let x be arbitrary such that $x \in A \cup B$; that is,
$(x \in A \vee x \in B)$

Case 1 $x \in A$: so $x \in X$ because $A \subseteq X$

Case 2 $x \in B$: so $x \in X$ because $B \subseteq X$

$\square$

Corollary 85 *Let* $\mathcal{U}$ *be a set and let* $A, B, C \in \mathcal{P}(\mathcal{U})$*.*

1. $C = A \cup B$

iff

$$[A \subseteq C \wedge B \subseteq C]$$

$$\wedge$$

$$[\forall X \in \mathcal{P}(\mathcal{U}).\ (A \subseteq X \ \wedge\ B \subseteq X) \implies C \subseteq X]$$

2. $C = A \cap B$

iff

$$[C \subseteq A \wedge C \subseteq B]$$

the intersection $A \cap B$ is the greatest (biggest) set with the property that is contained in both A & B.

$$\wedge$$

$$[\forall X \in \mathcal{P}(\mathcal{U}).\ (X \subseteq A \ \wedge\ X \subseteq B) \implies X \subseteq C]$$

Sets and logic

$\mathcal{P}(\mathbb{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathbb{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{c}$	$\neg(\cdot)$

Pairing axiom

For every a and b, there is a set with a and b as its only elements.

$$\{\, a\,,\ b \,\}$$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \ \vee \ x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{\, a \,\}$, and referred to as a *singleton*.

Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

Ordered pairing

unordered pairing of a and b

For every pair a and b, the set

$$\{\{a\},\{a,b\}\}$$

$\{a,b\} = \{b,a\}$

is abbreviated as

$$\langle a,b\rangle$$

$\neq \langle b,a\rangle$ for $a \neq b$

and referred to as an *ordered pair*.

Proposition 86 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff \big(a = x \wedge b = y\big) \quad .$$

PROOF: