# Denotational semantics of PCF

**Proposition.** *For all typing judgements* $\Gamma \vdash M : \tau$*, the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

*is a well-defined continous function.*

## Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{\perp\}$, we have

$$\boxed{[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

$$\mathcal{C}[\,] ::= [\,] \mid (\mathcal{C}[-])\,M \mid M\,(\mathcal{C}[-]) \mid \cdots \mid \mathrm{fix}(\mathcal{C}[-])$$

## Compositionality

**Proposition.** *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

*if* $[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

*then* $[\![\Gamma' \vdash \mathcal{C}[M]]\!] = [\![\Gamma' \vdash \mathcal{C}[M']]\!] : [\![\Gamma']\!] \to [\![\tau']\!]$

$M = M_1 M_2$

$$\frac{M_1 \Downarrow \mathbf{fn}\, x.M \qquad M[M_2/x] \Downarrow V}{M_1\, M_2 \Downarrow V}$$

Show $[\![M_1\, M_2]\!] \overset{?}{=} [\![V]\!]$

$= [\![M_1]\!]([\![M_2]\!])$

## Soundness

**Proposition.** *For all closed terms* $M, V \in \mathrm{PCF}_\tau$,

*if* $M \Downarrow_\tau V$ *then* $[\![M]\!] = [\![V]\!] \in [\![\tau]\!]$

$[\![M_1]\!] = [\![\mathbf{fn}\, x.M]\!]$ by ind

$\lambda d. [\![M]\!][x \mapsto d]$

$[\![M[M_2/x]]\!] = [\![V]\!]$

$[\![M_1\, M_2]\!] = [\![M]\!][x \mapsto [\![M_2]\!]] \overset{?}{=}$ Lemma

# Substitution property

**Proposition.** *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

*Then,*

$$
\begin{aligned}
&[\![\Gamma \vdash M'[M/x]]\!](\rho) \\
&\qquad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big)
\end{aligned}
$$

*for all* $\rho \in [\![\Gamma]\!]$.

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\end{aligned}
$$

*for all* $\rho \in [\![\Gamma]\!]$.

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In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau\rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau\rangle \vdash M']\!]([\![M]\!])$$

# *Topic 7*

Relating Denotational and Operational Semantics

## Adequacy

For any closed PCF terms $M$ and $V$ of *ground* type $\gamma \in \{nat, bool\}$ with $V$ a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V\,.$$

For $\gamma = nat$, $V = \underline{succ}^n(\underline{0})$ for some $n \in \mathbb{N}$. So $[\![V]\!] = n$

For $\gamma = bool$, $V = \underline{true}$ or $\underline{false}$ so $[\![V]\!] = true$ or $false$ (resp)

## Adequacy

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**NB**. Adequacy does not hold at function types

# Adequacy

For any closed PCF terms $M$ and $V$ of *ground* type $\gamma \in \{nat, bool\}$ with $V$ a value

$$\overline{V \Downarrow_\tau V}$$

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V\,.$$

**NB**. Adequacy does not hold at function types:

values

$$[\![\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x]\!] \;=\; [\![\mathbf{fn}\, x : \tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

$= \lambda d. [\![x:\tau \vdash (\mathrm{fn}\, y:\tau. y)\, x]\!][x \mapsto d]$ | $= \lambda d. [\![x:\tau \vdash x]\!][x \mapsto d]$

$= \lambda d. ([\![x:\tau \vdash \mathrm{fn}\, y:\tau. y]\!][x \mapsto d])([\![x:\tau \vdash x]\!][x \mapsto d]) = \ldots$

$= \lambda d. [x \mapsto d](x) = \lambda d. d = id$

$$[\![\Gamma \vdash x]\!]\rho = \rho(x)$$

$$\lambda d.\big([\![x:\tau \vdash \mathrm{fn}\, y:\tau.\, y]\!][x \mapsto d]\big)$$
$$\big([\![x:\tau \vdash x]\!][x \mapsto d]\big)$$

$$= \lambda d.\big(\lambda e.\, [\![x:\tau, y:\tau \vdash y]\!][x \mapsto d, y \mapsto e]\big)$$
$$\big([x \mapsto d](x)\big)$$

$$= \lambda d.\big(\lambda e.\, [x \mapsto d, y \mapsto e](y)\big)(d)\big)$$

$$= \lambda d.(\lambda e.e)\, d = \lambda d.d = id$$

## Adequacy

For any closed PCF terms $M$ and $V$ of *ground* type $\gamma \in \{nat, bool\}$ with $V$ a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V \ .$$

**NB**. Adequacy does not hold at function types:

$$[\![\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x]\!] \quad = \quad [\![\mathbf{fn}\, x : \tau.\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\mathbf{fn}\, x : \tau.\, (\mathbf{fn}\, y : \tau.\, y)\, x \not\Downarrow_{\tau \to \tau} \mathbf{fn}\, x : \tau.\, x$$

# Adequacy proof idea

## Adequacy proof idea

$[\![M]\!] = [\![V]\!] \Rightarrow M \Downarrow_\gamma V$

$\in [\![\gamma]\!]$

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider $M$ to be $M_1\, M_2$, $\mathbf{fix}(M')$.

Case $M = M_1 M_2$ ~ by ind on $M_1$ ~ $M_2$ ~

of higher type

Go on to prove a stronger statement that:

- it applies to all types in particular function types
- when instantiated at ground type implies adequacy.

# Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider $M$ to be $M_1\,M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

# Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider $M$ to be $M_1\, M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

   $$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   where the *formal approximation relations*

   a relation between denotations and closed terms.

   LOGICAL RELATIONS

   $$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   are *logically* chosen to allow a proof by induction.

   (defined by induction on types.

## Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{nat, bool\}$,

$$[\![M]\!] \lhd_\gamma M \text{ implies } \underbrace{\forall V \, ([\![M]\!] = [\![V]\!] \implies M \Downarrow_\gamma V)}_{\text{adequacy}}$$

$n \lhd_{nat} M \iff (n = \bot)$ or $(n \in \mathbb{N}$ and $M \Downarrow \underline{succ}^n(\underline{0}))$

$n \in \mathbb{N}_\bot$, $N \in PCF_{nat}$

## Definition of $d \lhd_\gamma M$ $(d \in [\![\gamma]\!], M \in \mathrm{PCF}_\gamma)$ for $\gamma \in \{nat, bool\}$

$$n \lhd_{nat} M \overset{\text{def}}{\Leftrightarrow} (n \in \mathbb{N} \Rightarrow M \Downarrow_{nat} \mathbf{succ}^n(\mathbf{0}))$$

$$\begin{aligned} b \lhd_{bool} M \overset{\text{def}}{\Leftrightarrow} & (b = \mathit{true} \Rightarrow M \Downarrow_{bool} \mathbf{true}) \\ & \& \ (b = \mathit{false} \Rightarrow M \Downarrow_{bool} \mathbf{false}) \end{aligned}$$

## Proof of: $[\![M]\!] \lhd_\gamma M$ implies **adequacy**

**Case** $\gamma = nat$.

$$
\begin{aligned}
& [\![M]\!] = [\![V]\!] \\
& \quad \Longrightarrow \; [\![M]\!] = [\![\mathbf{succ}^n(\mathbf{0})]\!] && \text{for some } n \in \mathbb{N} \\
& \quad \Longrightarrow \; n = [\![M]\!] \lhd_\gamma M \\
& \quad \Longrightarrow \; M \Downarrow \mathbf{succ}^n(\mathbf{0}) && \text{by definition of } \lhd_{nat}
\end{aligned}
$$

**Case** $\gamma = bool$ is similar.

$[\![M]\!] \lhd M$

$\rightsquigarrow$ by ind on $M$

## Requirements on the formal approximation relations, II

We want to be able to proceed by induction.

- Consider the case $M = M_1\, M_2$.

$[\![M_1\, M_2]\!] \overset{?}{\lhd}_{\tau} M_1(M_2)$

(∗) $\rightsquigarrow$ *logical* definition

$\|$

$[\![M_1]\!]([\![M_2]\!])$ $\rightsquigarrow$ by ind. $[\![M_1]\!] \lhd_{\sigma\to\tau} M_1$ and $[\![M_2]\!] \lhd_{\sigma} M_2$

Define $\lhd_{\sigma\to\tau}$ so that (∗); i.e. logically ...

## Definition of
## $f \lhd_{\tau \to \tau'} M \ \left(f \in (\llbracket \tau \rrbracket \to \llbracket \tau' \rrbracket), M \in \mathrm{PCF}_{\tau \to \tau'}\right)$

## Definition of $f \lhd_{\tau \to \tau'} M \ \big(f \in (\llbracket \tau \rrbracket \to \llbracket \tau' \rrbracket), M \in \mathrm{PCF}_{\tau \to \tau'}\big)$

$$f \lhd_{\tau \to \tau'} M$$

$$\overset{\text{def}}{\Leftrightarrow} \quad \forall\, x \in \llbracket \tau \rrbracket, N \in \mathrm{PCF}_\tau$$

$$(x \lhd_\tau N \ \Rightarrow \ f(x) \lhd_{\tau'} M\, N)$$

$$\mathrm{fix}\,[\![M]\!]$$
$$\parallel$$
$$[\![\mathrm{fix}(M')]\!] \lhd_\tau \mathrm{fix}(M')$$

$\rightsquigarrow$ want to prove a property of a fix

## Requirements on the formal approximation relations, III

We want to be able to proceed by induction. let us use Scott induction

- Consider the case $M = \mathbf{fix}(M')$.

$\rightsquigarrow$ *admissibility* property

need

Check

$$\{x \mid x \lhd_\tau N\} \subseteq [\![\tau]\!]$$

is admissible.

## Admissibility property

**Lemma.** *For all types $\tau$ and $M \in \mathrm{PCF}_\tau$, the set*

$$\{ d \in [\![\tau]\!] \mid d \lhd_\tau M \}$$

*is an admissible subset of $[\![\tau]\!]$.*

$$\left[\; d \triangleleft_{\tau} \mathrm{fix}(M) \;\right]$$

By ind

$$[\![M]\!] \triangleleft_{\tau \to \tau} M$$

$$[\![M]\!](d) \triangleleft M(\mathrm{fix}\, M)$$

$$\|$$

$$[\![M]\!]\, d \triangleleft \mathrm{fix}(M)$$

$$\frac{M(\mathrm{fix}\, M) \Downarrow V}{\mathrm{fix}\, M \Downarrow V}$$

$\{$ gap

$$\frac{x \triangleleft N, \quad N \Downarrow V \Rightarrow M \Downarrow V}{x \triangleleft M}$$

$$\frac{d \triangleleft \mathrm{fix}(M) \Rightarrow [\![M]\!]\, d \triangleleft \mathrm{fix}(M)}{\mathrm{fix}\, [\![M]\!] = [\![\mathrm{fix}\, M]\!] \triangleleft \mathrm{fix}(M)}$$

## Further properties

**Lemma.** *For all types* $\tau$*, elements* $d, d' \in [\![\tau]\!]$*, and terms* $M, N, V \in \mathrm{PCF}_\tau$*,*

1. *If* $d \sqsubseteq d'$ *and* $d' \lhd_\tau M$ *then* $d \lhd_\tau M$*.*

2. *If* $d \lhd_\tau M$ *and* $\forall V\,(M \Downarrow_\tau V \implies N \Downarrow_\tau V)$ *then* $d \lhd_\tau N$ *.*

## Requirements on the formal approximation relations, IV

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fn}\, x : \tau \,.\, M'$.

$\rightsquigarrow$ *substitutivity* property for open terms

## Fundamental property

**Theorem.** *For all* $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ *and all* $\Gamma \vdash M : \tau$, *if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_\tau M[M_1/x_1, \ldots, M_n/x_n]$ .

## Fundamental property

**Theorem.** *For all* $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ *and all* $\Gamma \vdash M : \tau$, *if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_\tau M[M_1/x_1, \ldots, M_n/x_n]$ *.*

**NB.** The case $\Gamma = \emptyset$ reduces to

$$[\![M]\!] \lhd_\tau M$$

for all $M \in \mathrm{PCF}_\tau$.

## Fundamental property of the relations $\lhd_\tau$

**Proposition.** *If* $\Gamma \vdash M : \tau$ *is a valid PCF typing, then for all* $\Gamma$*-environments* $\rho$ *and all* $\Gamma$*-substitutions* $\sigma$

$$\rho \lhd_\Gamma \sigma \;\Rightarrow\; [\![\Gamma \vdash M]\!](\rho) \lhd_\tau M[\sigma]$$

- $\rho \lhd_\Gamma \sigma$ means that $\rho(x) \lhd_{\Gamma(x)} \sigma(x)$ holds for each $x \in dom(\Gamma)$.
- $M[\sigma]$ is the PCF term resulting from the simultaneous substitution of $\sigma(x)$ for $x$ in $M$, each $x \in dom(\Gamma)$.