Complexity Theory
Lecture 4

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http://www.cl.cam.ac.uk/teaching/1516/Complexity/

Composites

Consider the decision problem (or *language*) Composite defined by:

$$\{x \mid x \text{ is not prime}\}$$

This is the complement of the language Prime.

Is Composite $\in$ P?

Clearly, the answer is yes if, and only if, Prime $\in$ P.

Satisfiability

For Boolean expressions ϕ that contain variables, we can ask

> Is there an assignment of truth values to the variables which would make the formula evaluate to `true`?

The set of Boolean expressions for which this is true is the language SAT of *satisfiable* expressions.

This can be decided by a deterministic Turing machine in time $O(n^2 2^n)$.

An expression of length n can contain at most n variables.

For each of the 2^n possible truth assignments to these variables, we check whether it results in a Boolean expression that evaluates to `true`.

Is SAT $\in$ P?

Hamiltonian Graphs

Given a graph $G = (V, E)$, a *Hamiltonian cycle* in G is a path in the graph, starting and ending at the same node, such that every node in V appears on the cycle *exactly once*.

A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The language HAM is the set of encodings of Hamiltonian graphs.

Is HAM $\in$ P?

Graph Isomorphism

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is there a *bijection*

$$\iota : V_1 \to V_2$$

such that for every $u, v \in V_1$,

$$(u, v) \in E_1 \quad \text{if, and only if,} \quad (\iota(u), \iota(v)) \in E_2.$$

Is Graph Isomorphism $\in \mathsf{P}$?

Polynomial Verification

The problems Composite, SAT, HAM and Graph Isomorphism have something in common.

In each case, there is a *search space* of possible solutions.

> the numbers less than x; truth assignments to the variables of ϕ; lists of the vertices of G; a bijection between V_1 and V_2.

The size of the search is *exponential* in the length of the input.

Given a potential solution in the search space, it is *easy* to check whether or not it is a solution.

Verifiers

A verifier V for a language L is an algorithm such that

$$L = \{x \mid (x, c) \text{ is accepted by } V \text{ for some } c\}$$

If V runs in time polynomial in the length of x, then we say that

L is *polynomially verifiable*.

Many natural examples arise, whenever we have to construct a solution to some design constraints or specifications.

Nondeterminism

If, in the definition of a Turing machine, we relax the condition on δ being a function and instead allow an arbitrary relation, we obtain a *nondeterministic Turing machine*.

$$\delta \subseteq (Q \times \Sigma) \times (Q \cup \{\mathrm{acc}, \mathrm{rej}\} \times \Sigma \times \{R, L, S\}).$$

The yields relation $\rightarrow_M$ is also no longer functional.

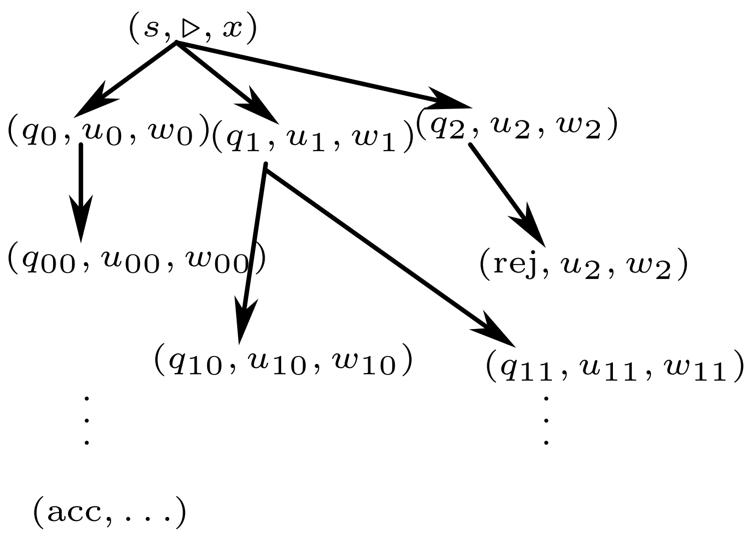
We still define the language accepted by M by:

$$\{x \mid (s, \triangleright, x) \rightarrow_M^\star (\mathrm{acc}, w, u) \text{ for some } w \text{ and } u\}$$

though, for some x, there may be computations leading to accepting as well as rejecting states.

Computation Trees

With a nondeterministic machine, each configuration gives rise to a tree of successive configurations.



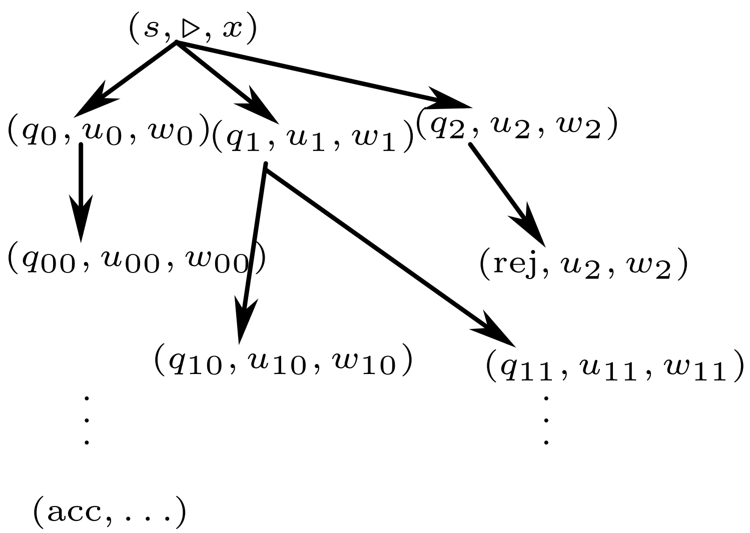
Nondeterministic Complexity Classes

We have already defined $\mathsf{TIME}(f)$ and $\mathsf{SPACE}(f)$.

$\mathsf{NTIME}(f)$ is defined as the class of those languages L which are accepted by a *nondeterministic* Turing machine M, such that for every $x \in L$, there is an accepting computation of M on x of length at most $f(n)$, where n is the length of x.

$$\mathsf{NP} = \bigcup_{k=1}^{\infty} \mathsf{NTIME}(n^k)$$

Nondeterminism



For a language in NTIME(f), the height of the tree can be bounded by $f(n)$ when the input is of length n.

NP

A language L is polynomially verifiable if, and only if, it is in NP.

To prove this, suppose L is a language, which has a verifier V, which runs in time $p(n)$.

The following describes a *nondeterministic algorithm* that accepts L

1. input x of length n
2. nondeterministically guess c of length $\leq p(n)$
3. run V on (x, c)

NP

In the other direction, suppose M is a nondeterministic machine that accepts a language L in time n^k.

We define the *deterministic algorithm* V which on input (x, c) simulates M on input x.

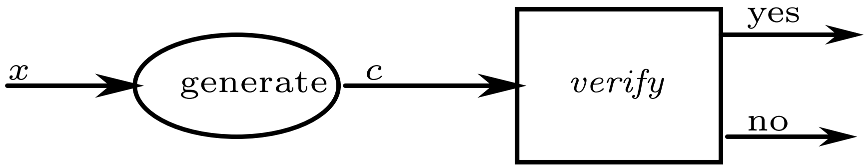
At the i^{th} nondeterministic choice point, V looks at the i^{th} character in c to decide which branch to follow.

If M accepts then V accepts, otherwise it rejects.

V is a polynomial verifier for L.

Generate and Test

We can think of nondeterministic algorithms in the generate-and test paradigm:



Where the *generate* component is nondeterministic and the *verify* component is deterministic.

Reductions

Given two languages $L_1 \subseteq \Sigma_1^\star$, and $L_2 \subseteq \Sigma_2^\star$,

A *reduction* of L_1 to L_2 is a *computable* function

$$f : \Sigma_1^\star \to \Sigma_2^\star$$

such that for every string $x \in \Sigma_1^\star$,

$$f(x) \in L_2 \text{ if, and only if, } x \in L_1$$

Resource Bounded Reductions

If f is computable by a polynomial time algorithm, we say that L_1 is *polynomial time reducible* to L_2.

$$L_1 \leq_P L_2$$

If f is also computable in $\mathsf{SPACE}(\log n)$, we write

$$L_1 \leq_L L_2$$

Reductions 2

If $L_1 \leq_P L_2$ we understand that L_1 is no more difficult to solve than L_2, at least as far as polynomial time computation is concerned.

That is to say,

> If $L_1 \leq_P L_2$ and $L_2 \in \mathsf{P}$, then $L_1 \in \mathsf{P}$

We can get an algorithm to decide L_1 by first computing f, and then using the polynomial time algorithm for L_2.

Completeness

The usefulness of reductions is that they allow us to establish the *relative* complexity of problems, even when we cannot prove absolute lower bounds.

Cook (1972) first showed that there are problems in NP that are maximally difficult.

A language L is said to be NP-*hard* if for every language $A \in \mathsf{NP}$, $A \leq_P L$.

A language L is NP-*complete* if it is in NP and it is NP-hard.