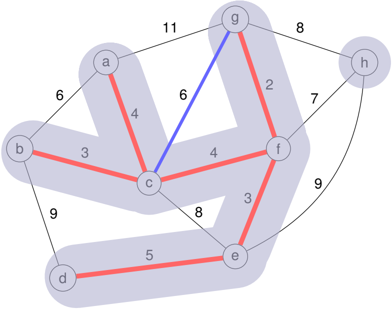

6.3: Minimum Spanning Tree

Frank Stajano Thomas Sauerwald

Lent 2016

UNIVERSITY OF CAMBRIDGE

Minimum Spanning Tree Problem

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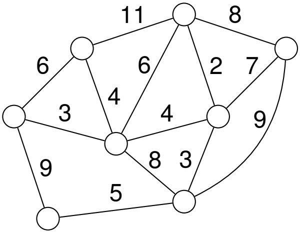
- Given: undirected, connected graph $G = (V, E, w)$ with non-negative edge weights



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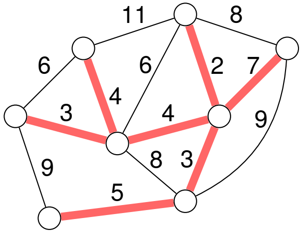

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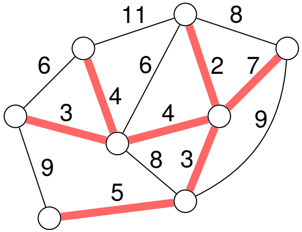
Must be necessarily a tree!



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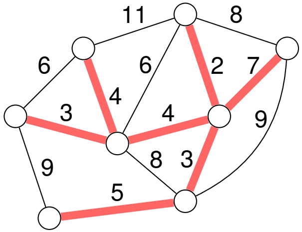

Applications

- Street Networks, Wiring Electronic Components, Laying Pipes
- Weights may represent distances, costs, travel times, capacities, resistance etc.

Generic Algorithm

```
def minimum spanningTree(G)
  A = empty set of edges
  while A does not span all vertices yet:
    add a safe edge to A
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How to find a safe edge?

Finding safe edges

Definitions

- a cut is a partition of V into at least two disjoint sets

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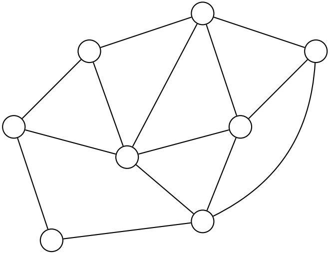

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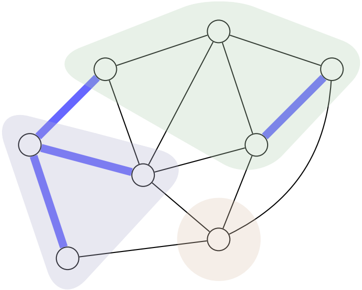

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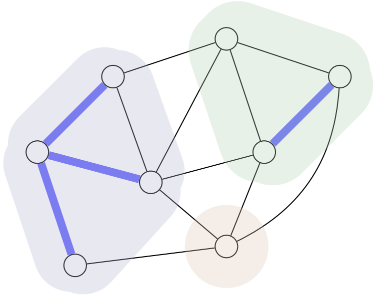

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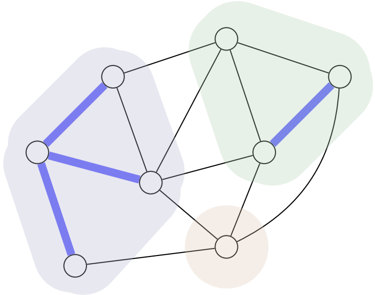

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Let $A \subseteq E$ be a subset of a MST of G. Then for any cut that respects A, the lightest edge of G that goes across the cut is safe.

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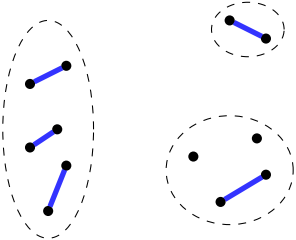

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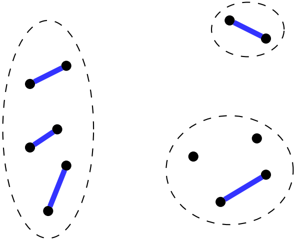

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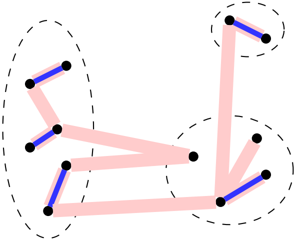

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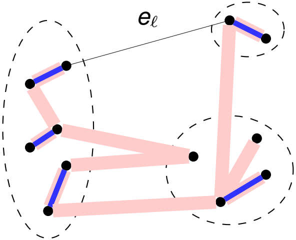

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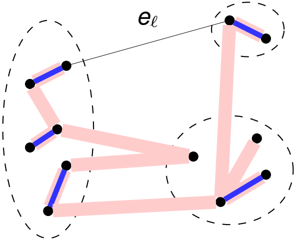

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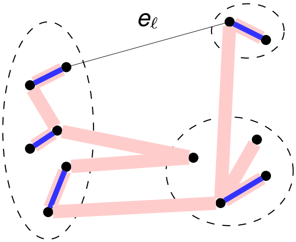

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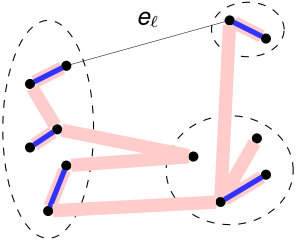

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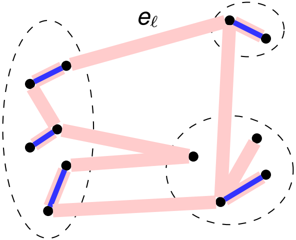

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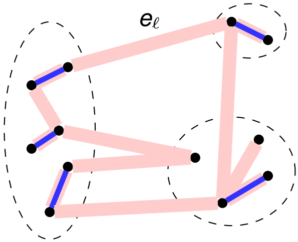

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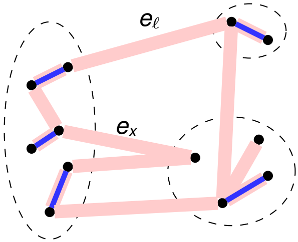

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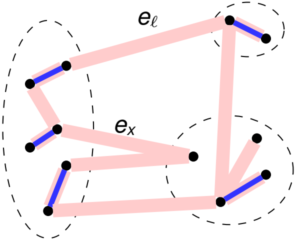

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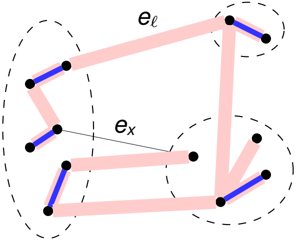

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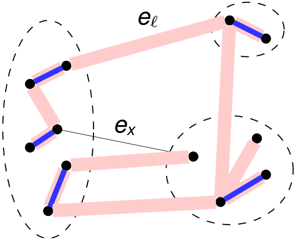

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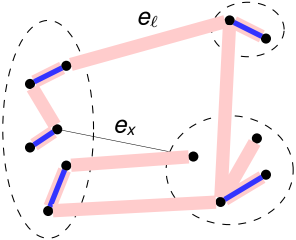

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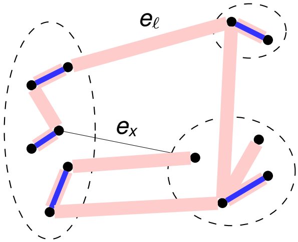
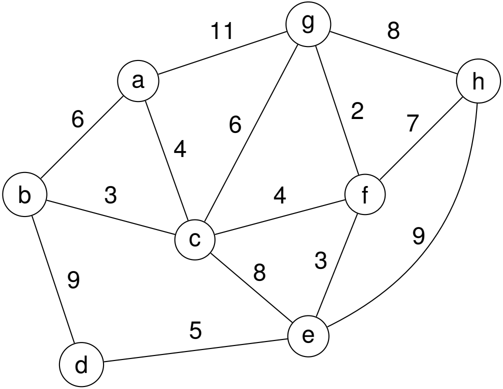


Glimpse at Kruskal's Algorithm

Basic Strategy



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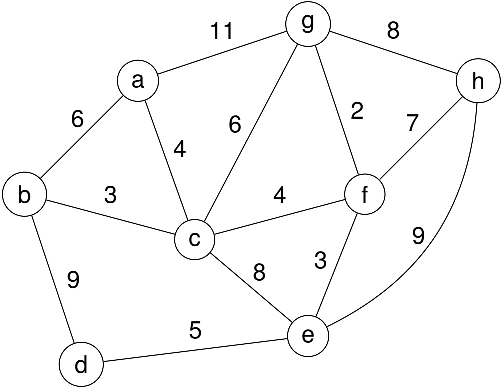

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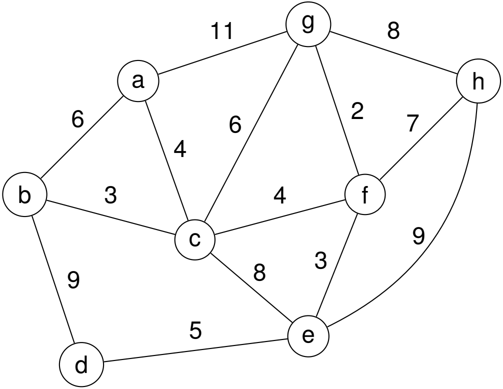

Glimpse at Kruskal’s Algorithm

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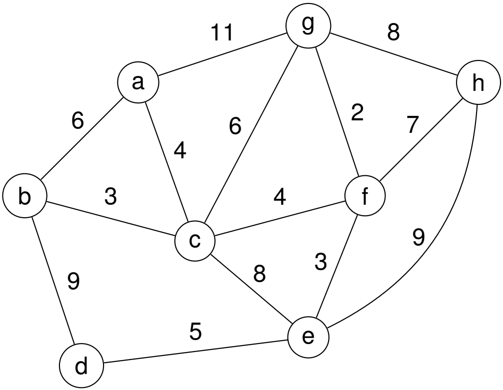

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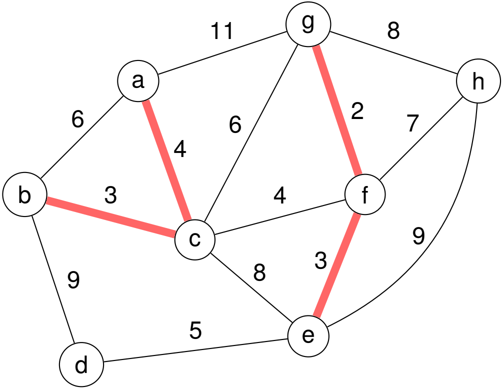

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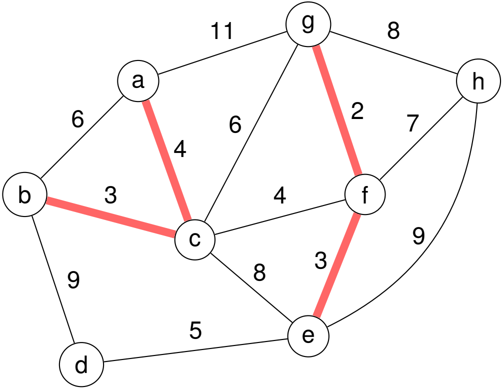

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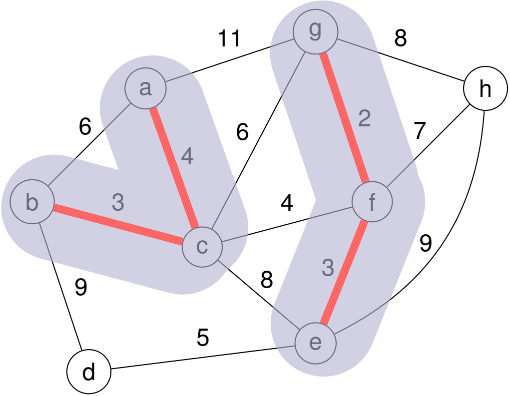

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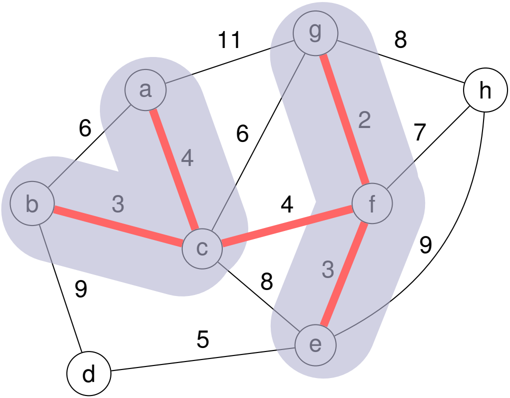

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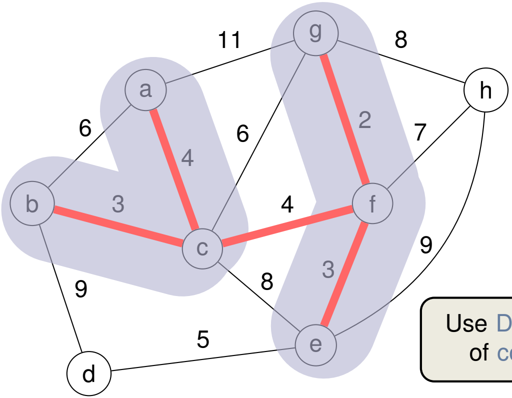

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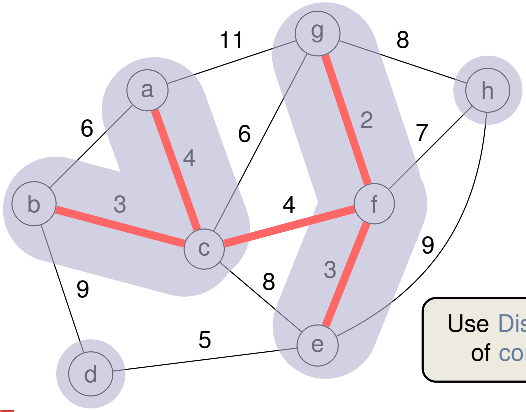

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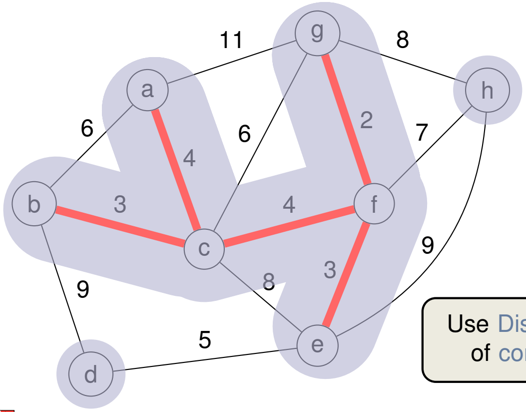

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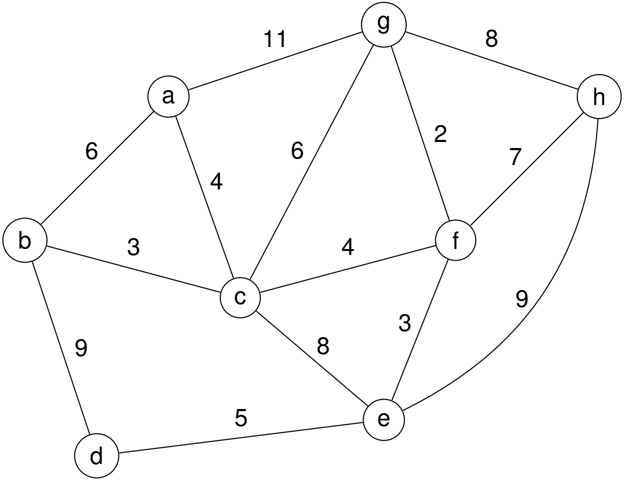

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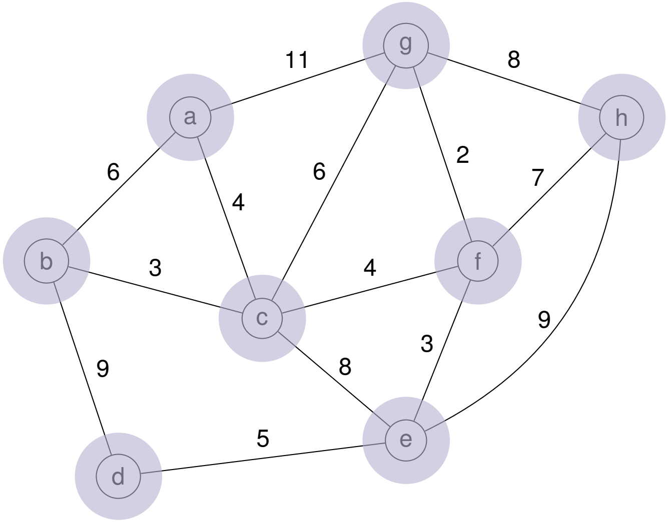

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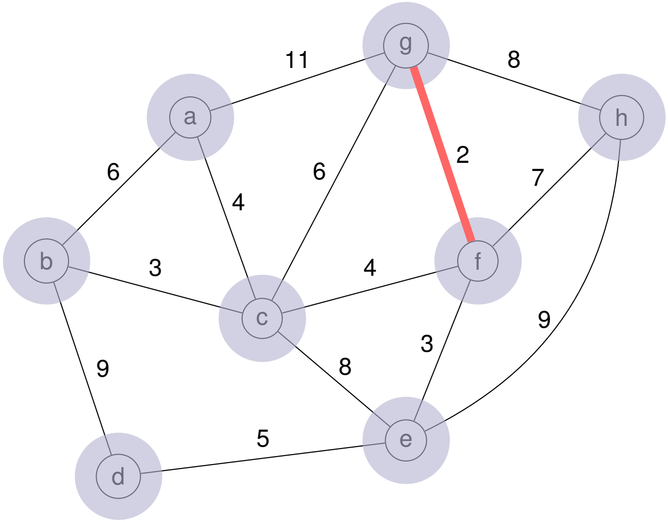

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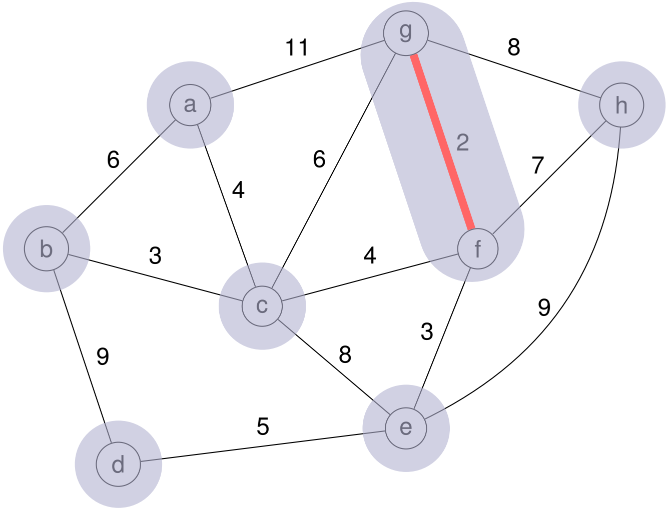

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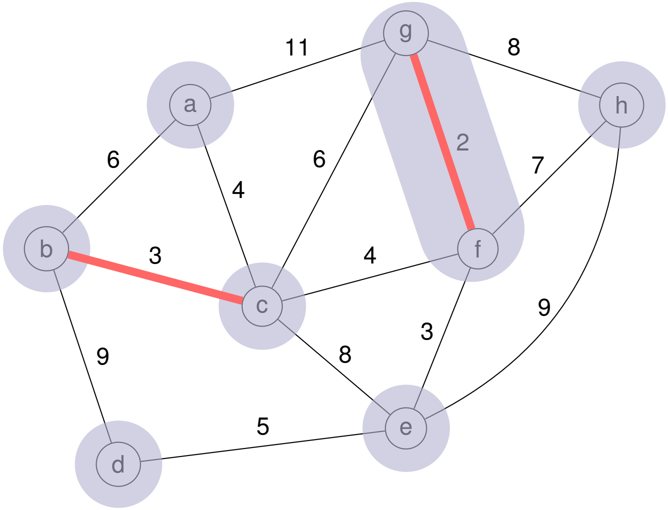

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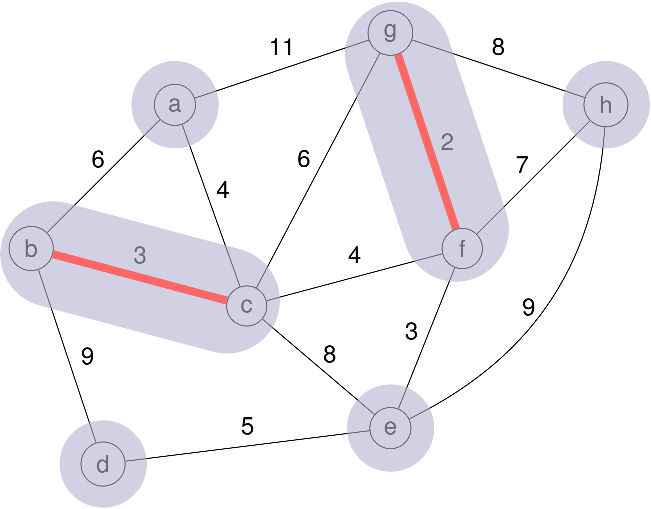

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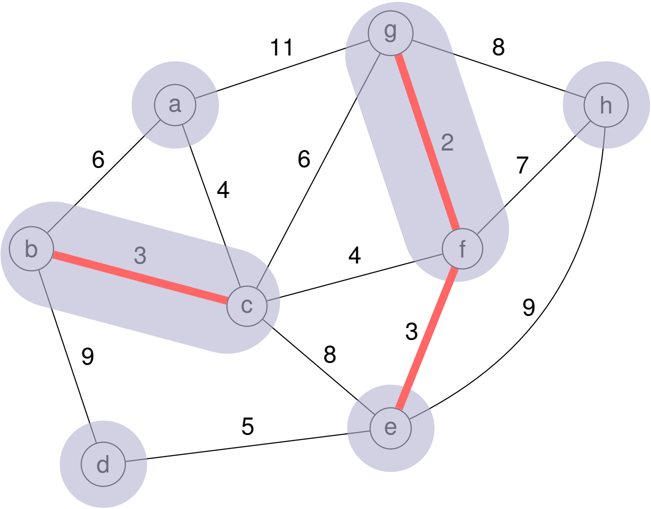

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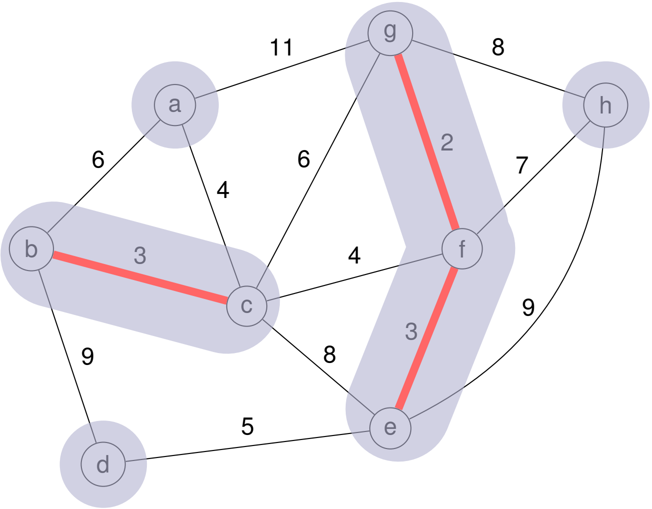

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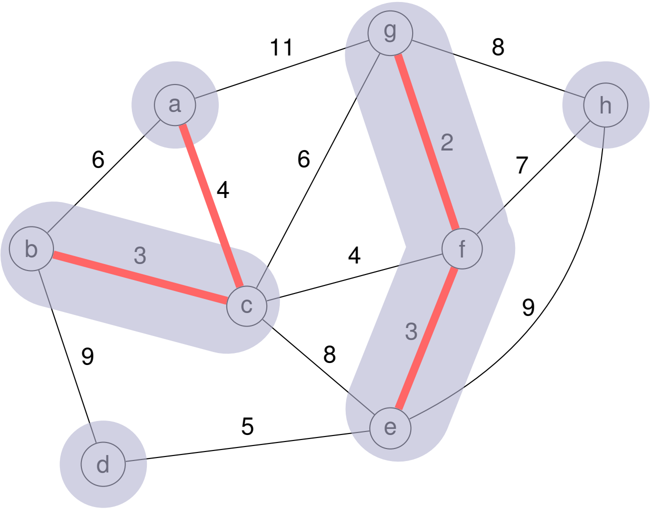

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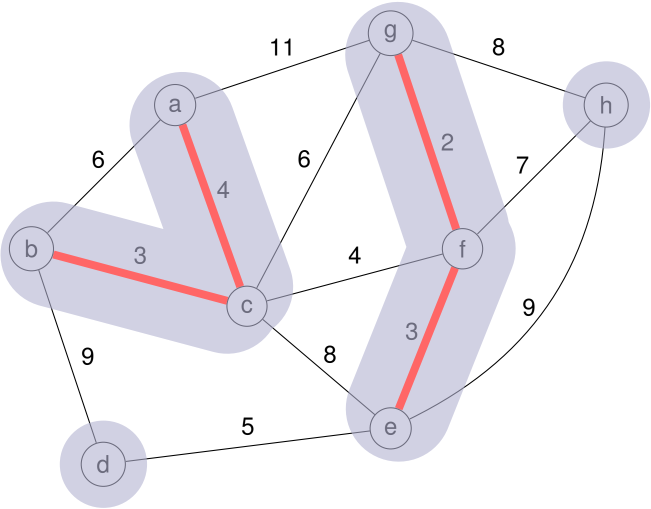

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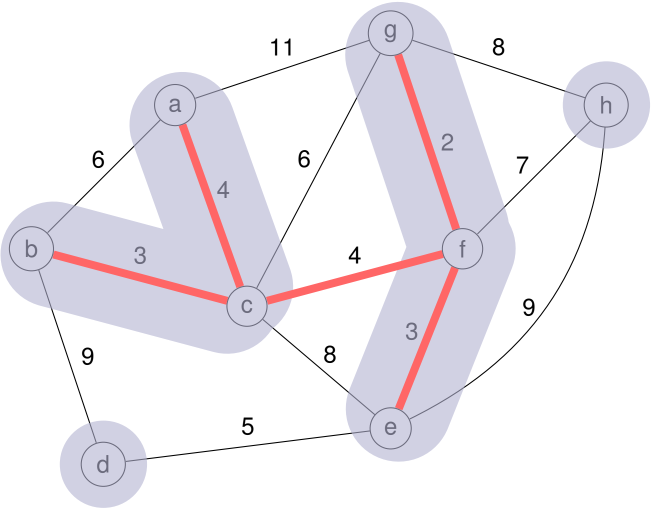

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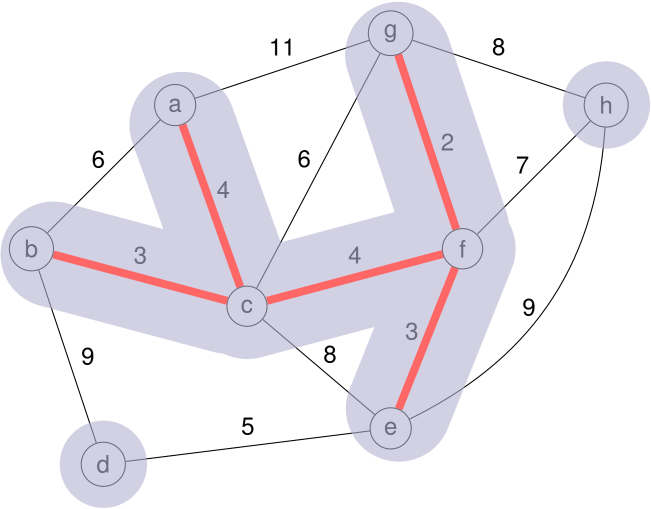

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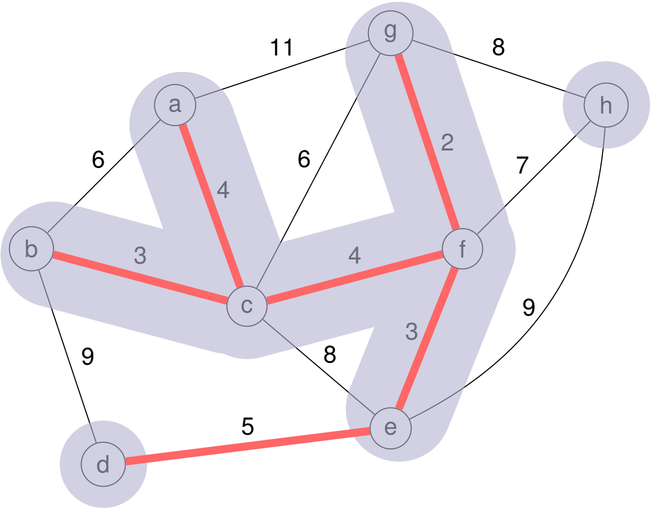

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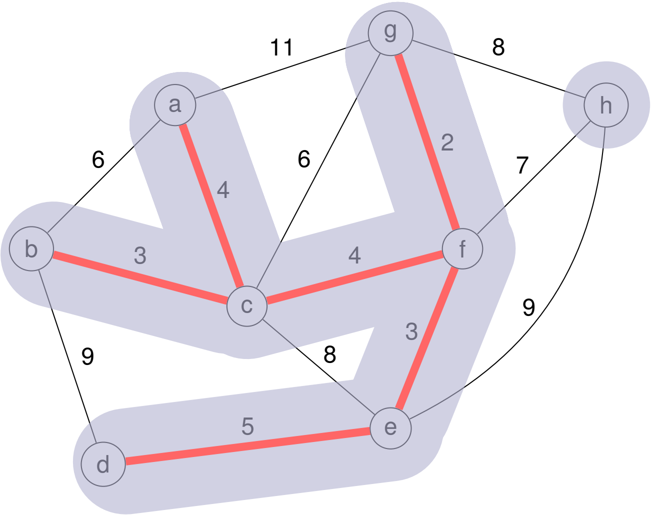

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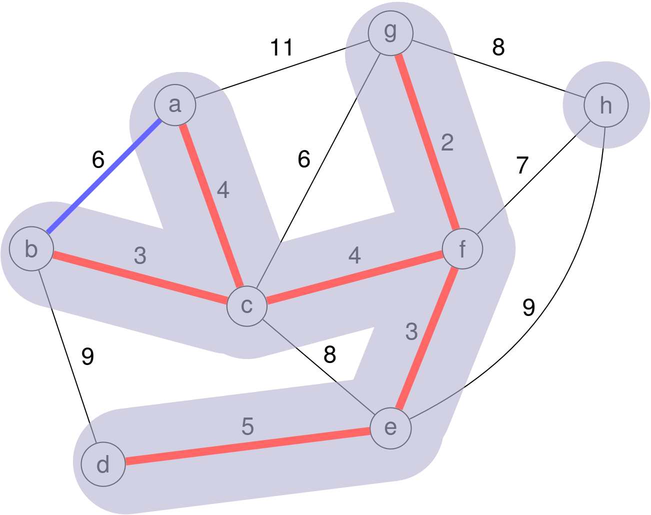

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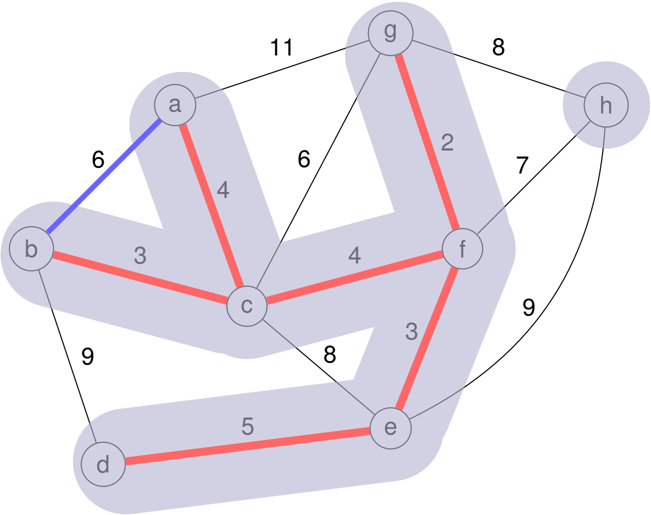

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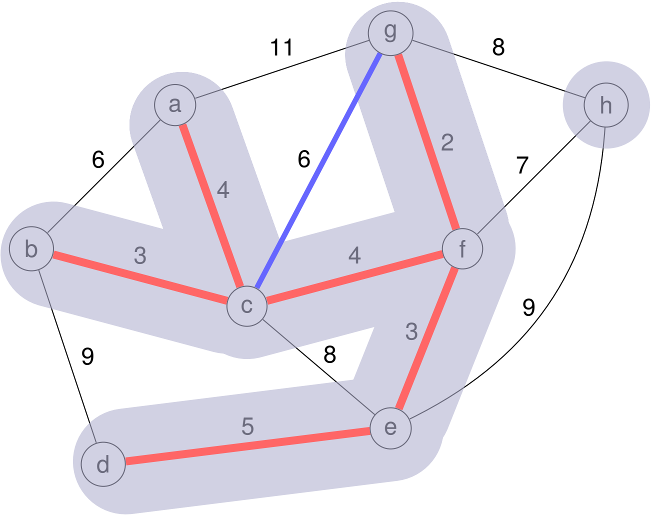

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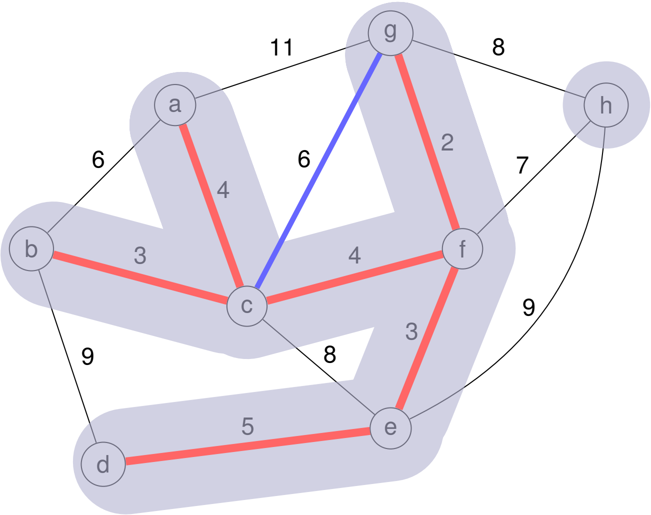

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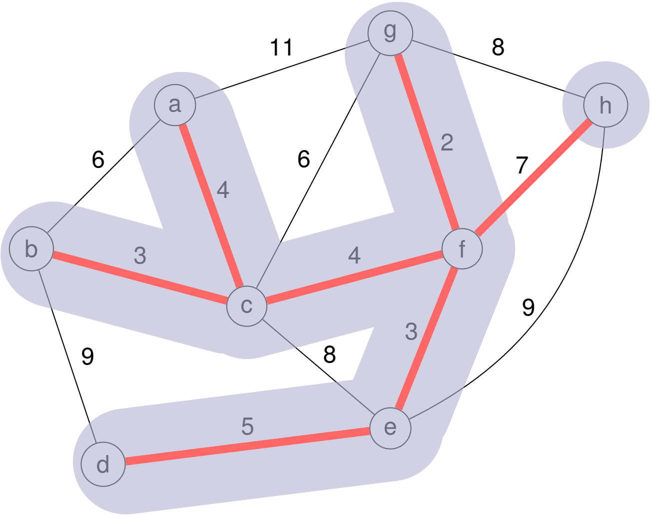

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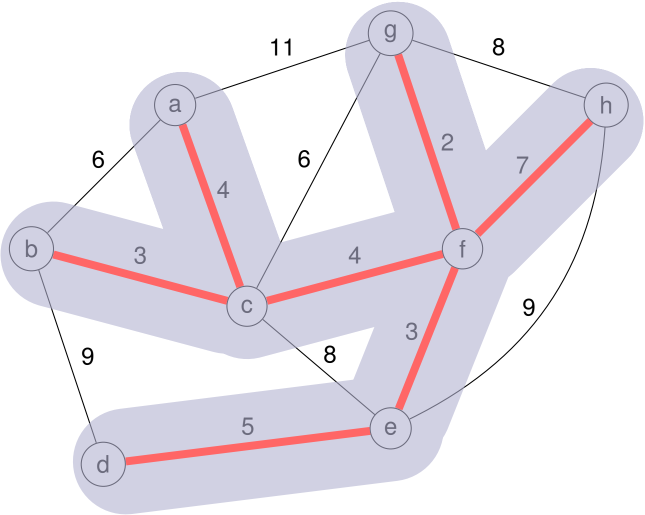

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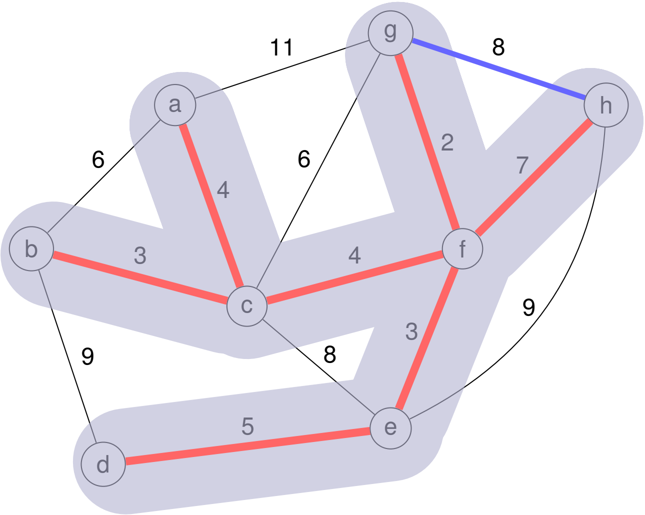

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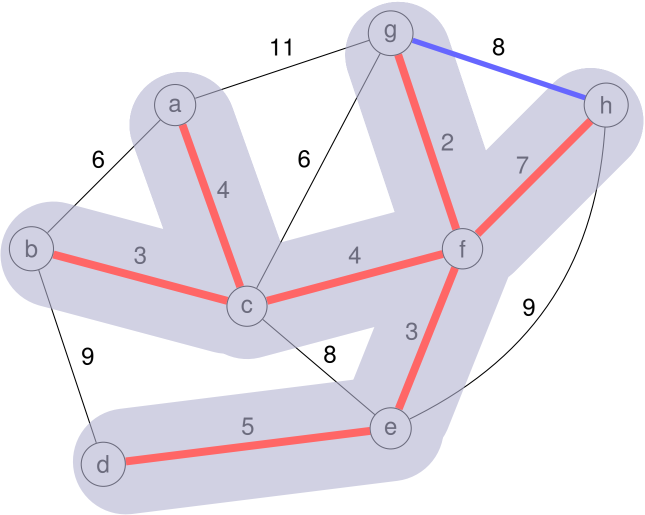

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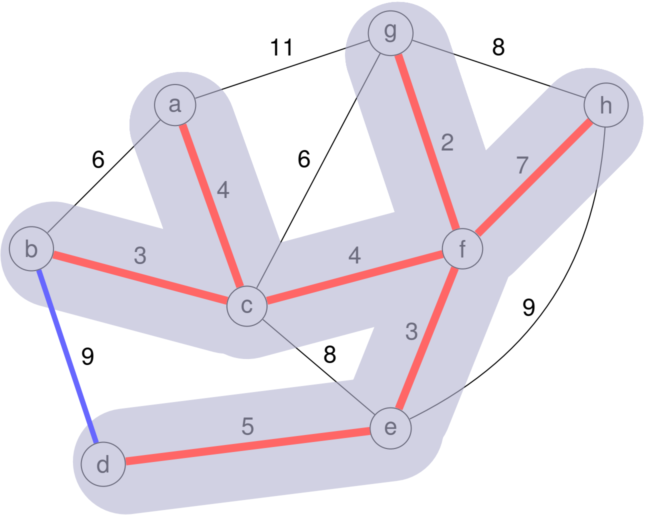

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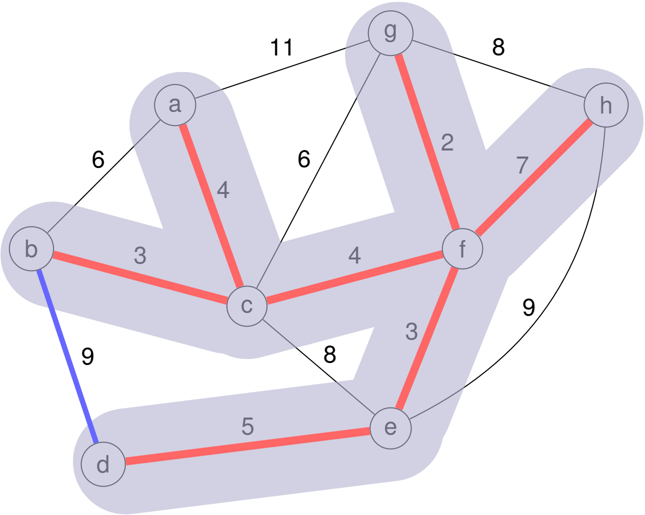

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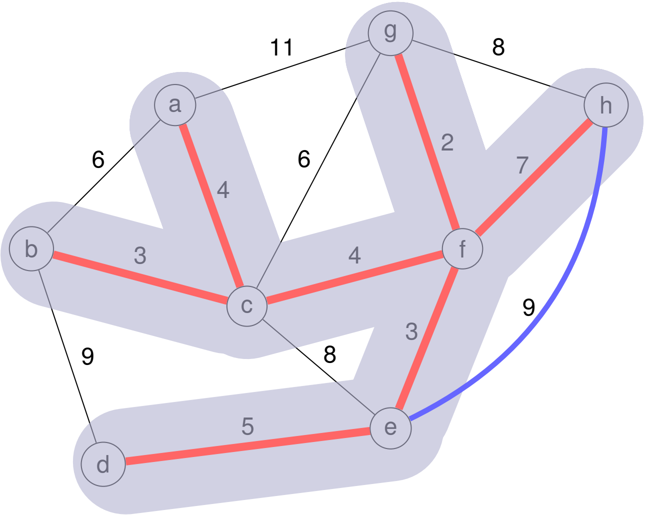

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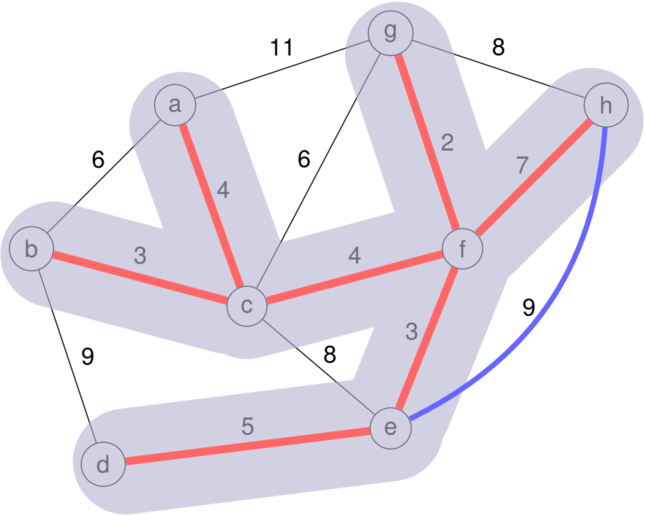

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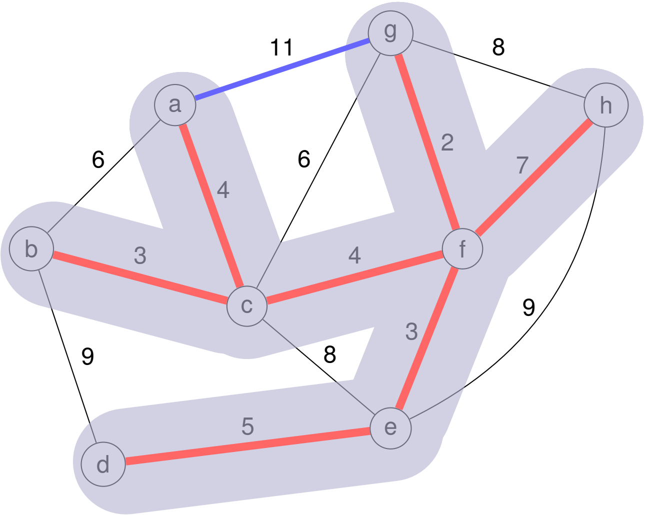

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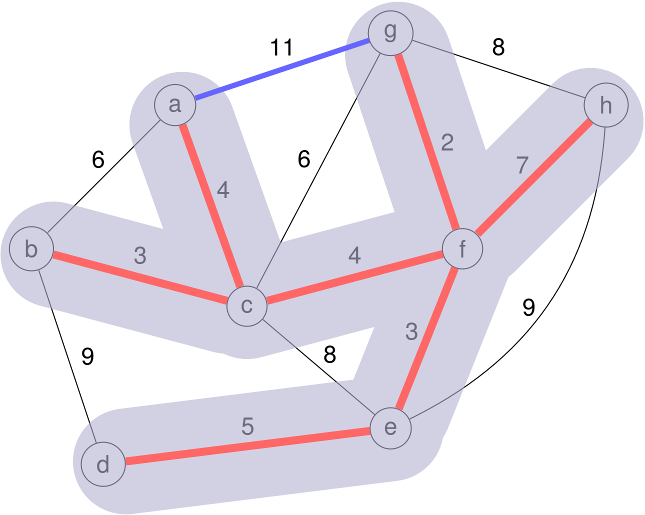

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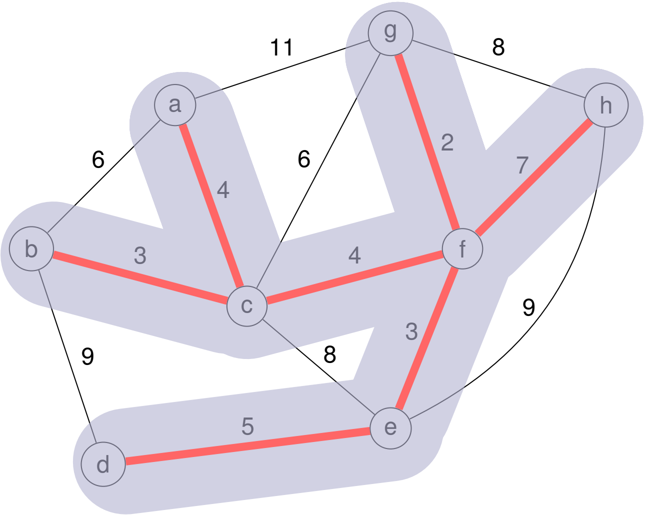

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If edges are already sorted, runtime becomes $O(E \cdot \alpha(n))$!

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- Consider the cut of all connected components (disjoint sets)

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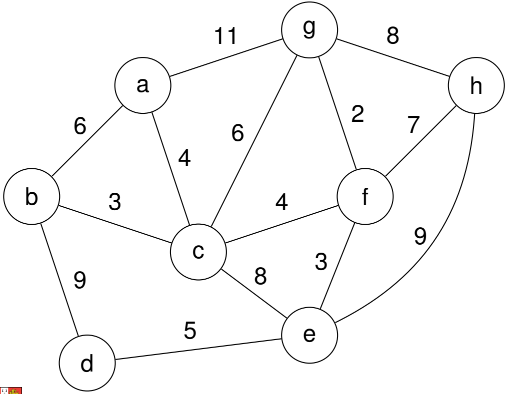
Correctness

- Consider the cut of all connected components (disjoint sets)
- L. 14 ensures that we extend A by an edge that goes across the cut
- This edge is also the lightest edge crossing the cut (otherwise, we would have included a lighter edge before)

Prim's Algorithm

Basic Strategy

- Start growing a tree from a designated root vertex



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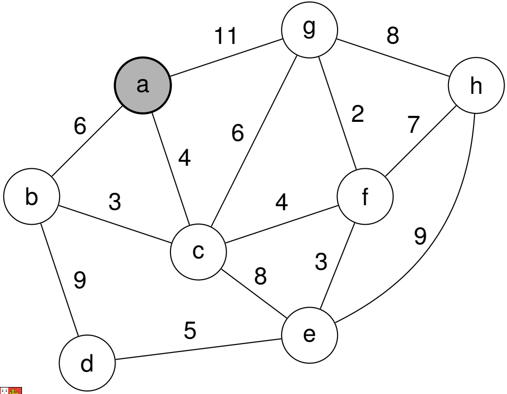

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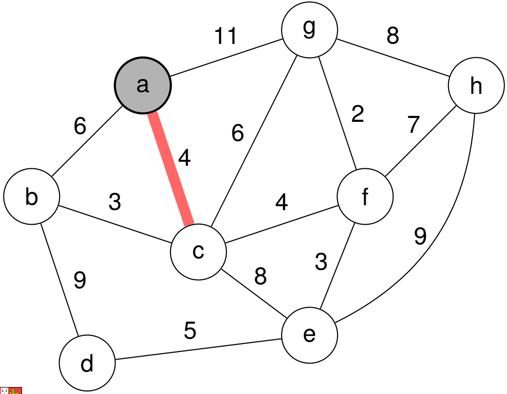

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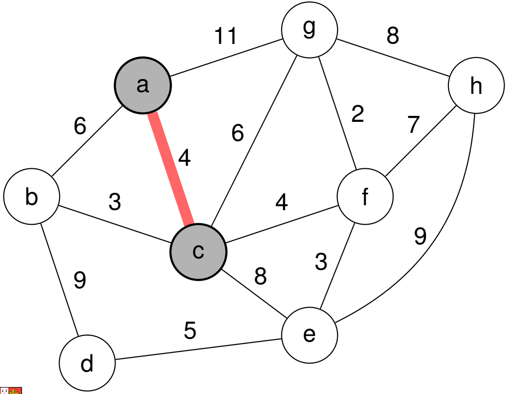

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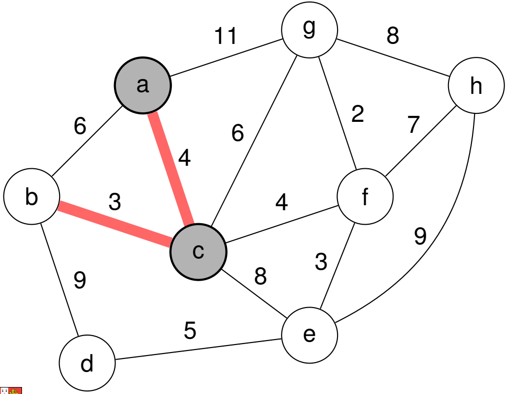

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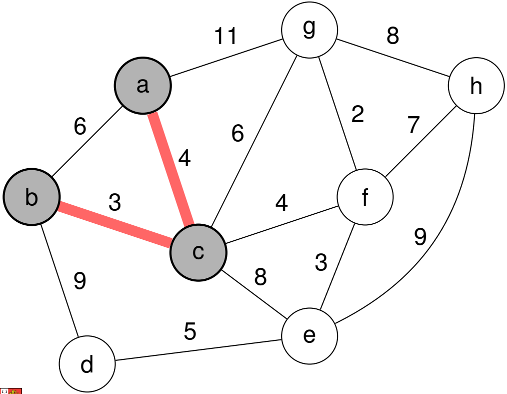

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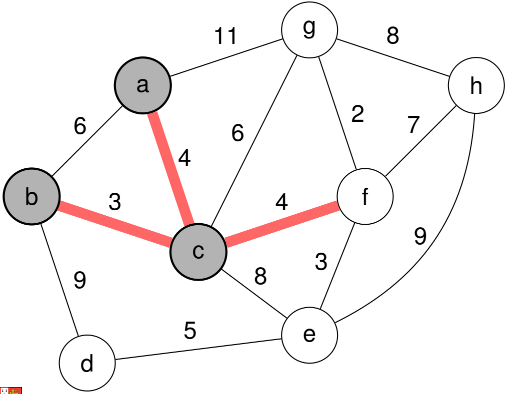

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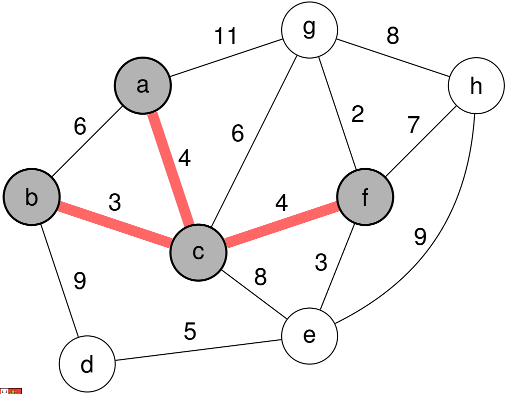

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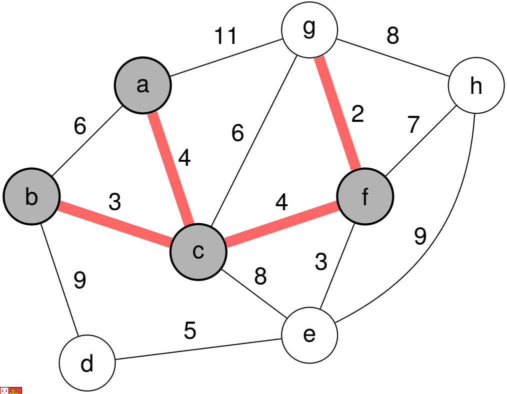

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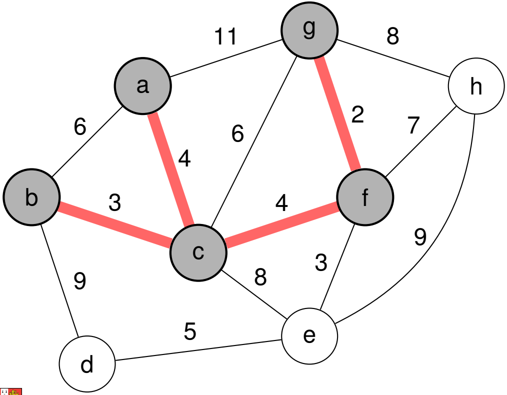

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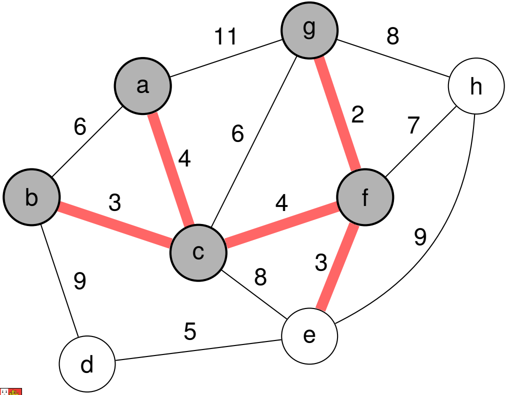

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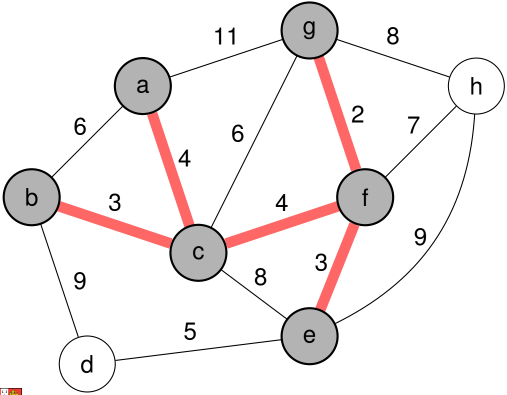

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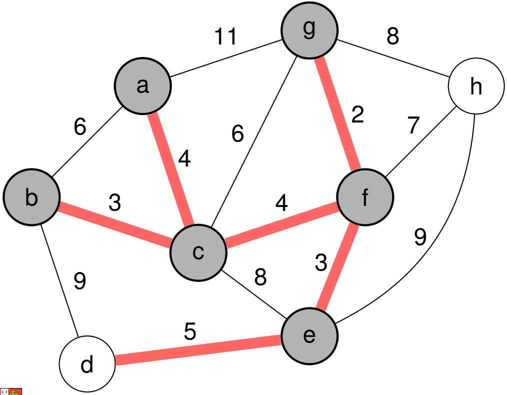

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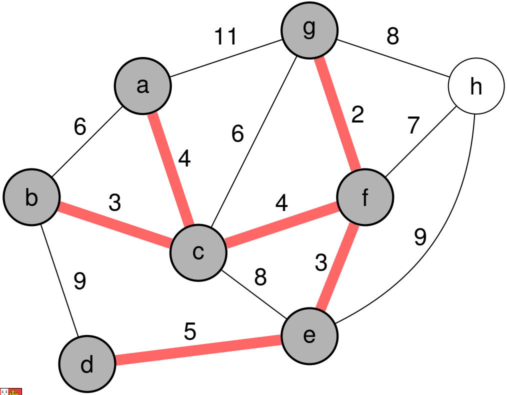

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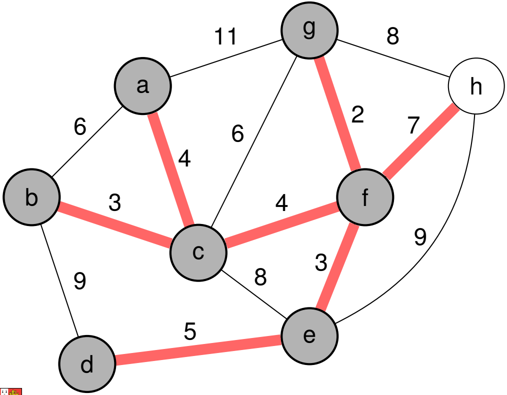

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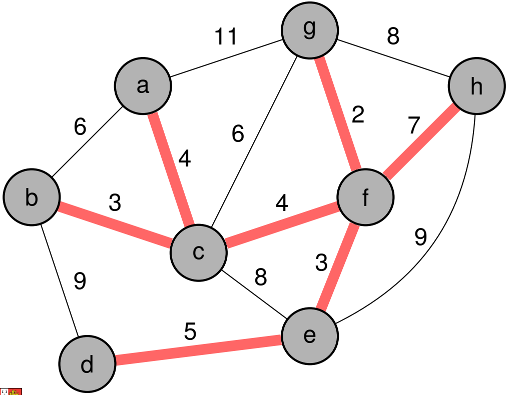

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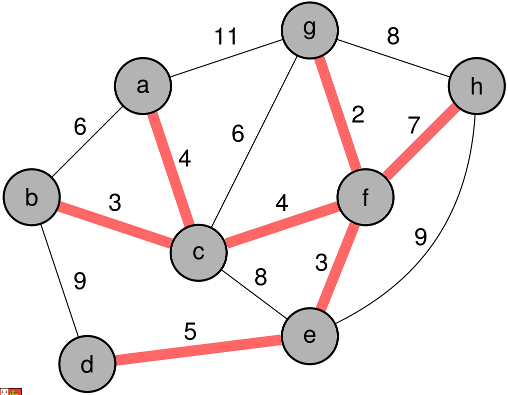
Implementation will be based on vertices!



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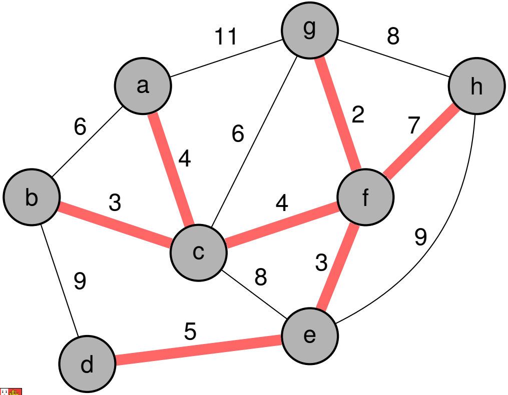
Assign every vertex not in A a key which is at all stages equal to the smallest weight of an edge connecting to A



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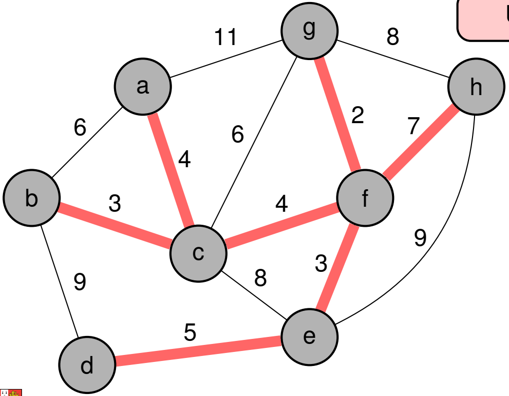
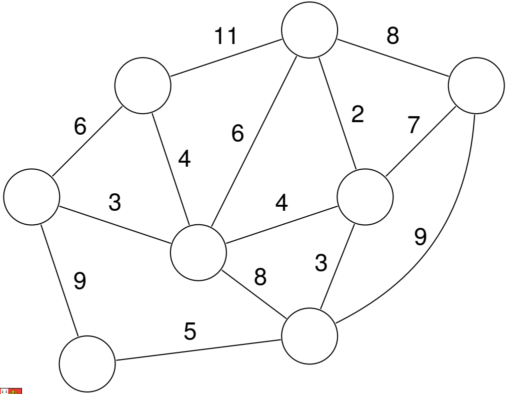
Use a Priority Queue!



Prim's Algorithm

Implementation

- Every vertex in Q has key and pointer of least-weight edge to $V \setminus Q$
- At each step:
  1. extract vertex from Q with smallest key $\Leftrightarrow$ safe edge of cut $(V \setminus Q, Q)$
  2. update keys and pointers of its neighbors in Q



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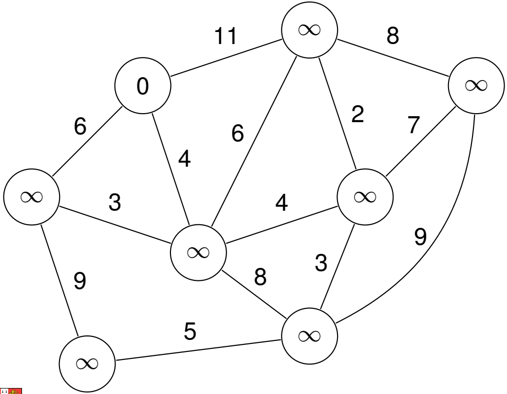

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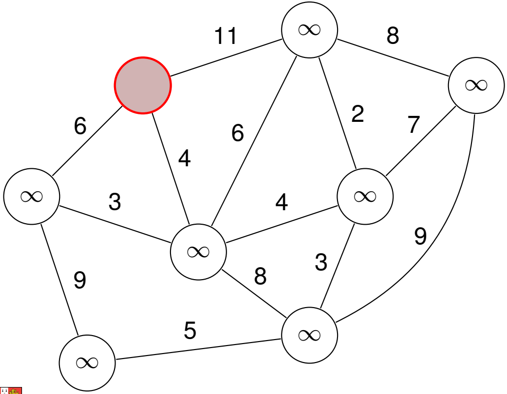

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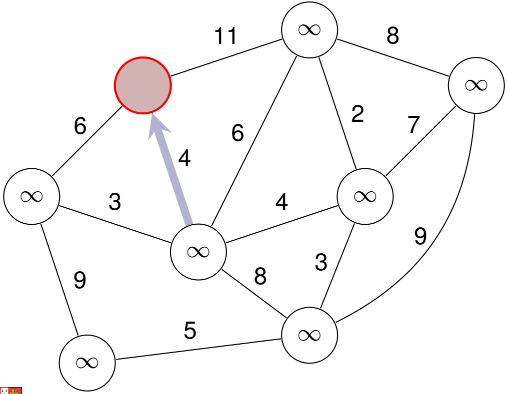

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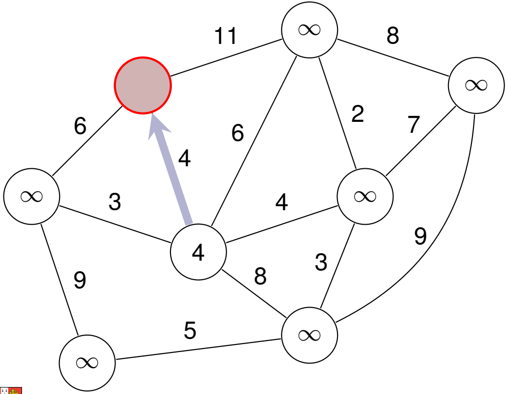

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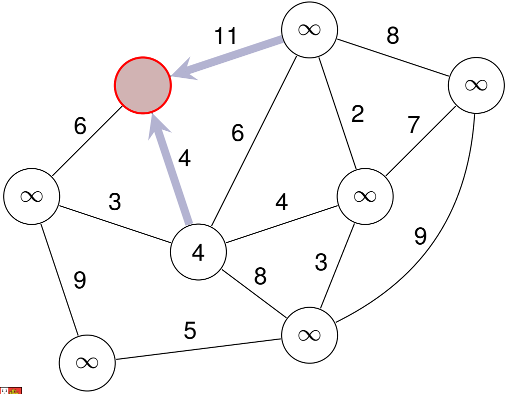

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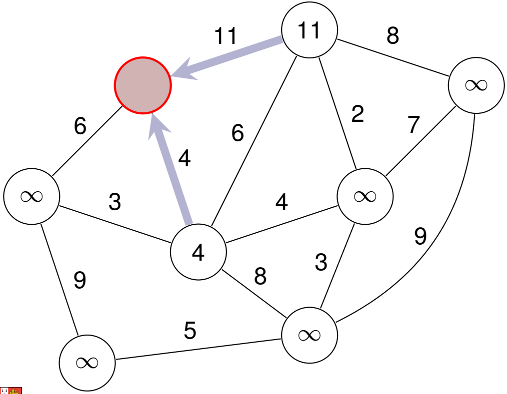

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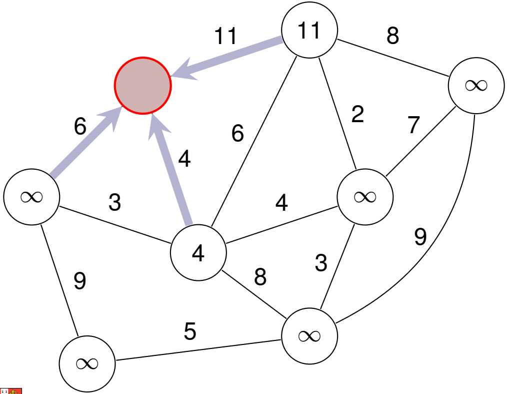

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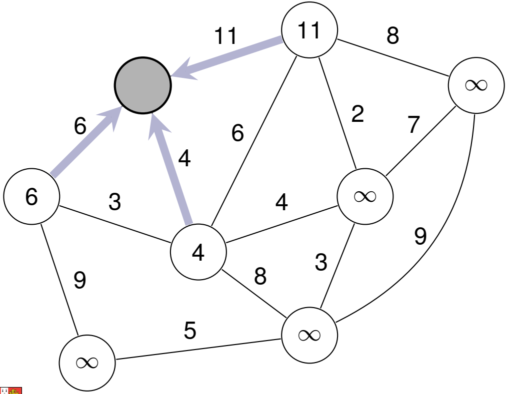

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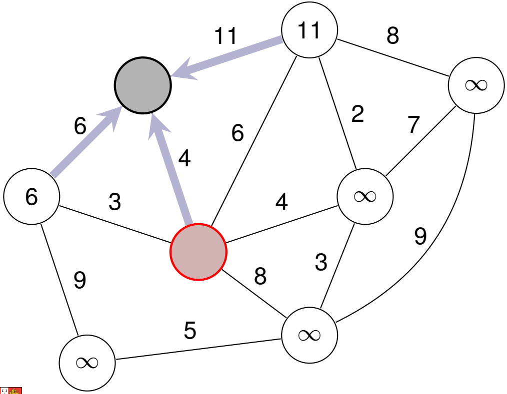

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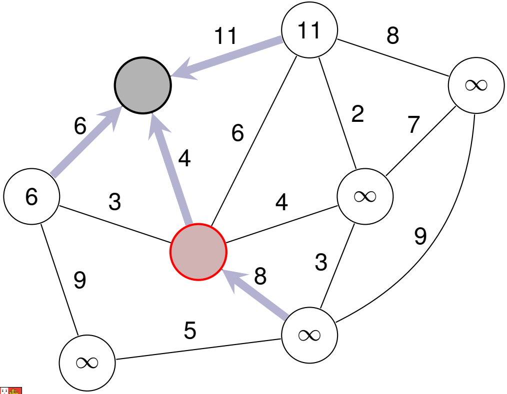

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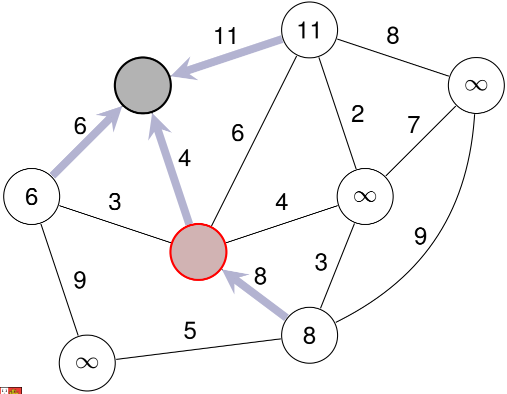

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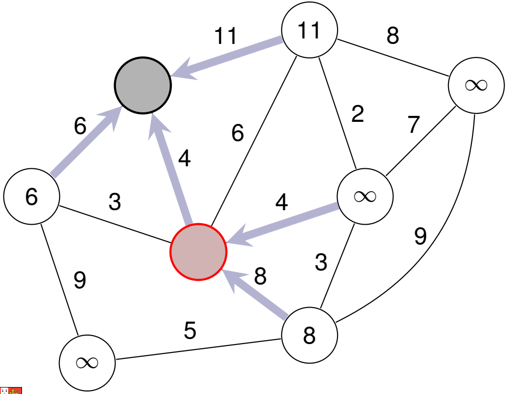

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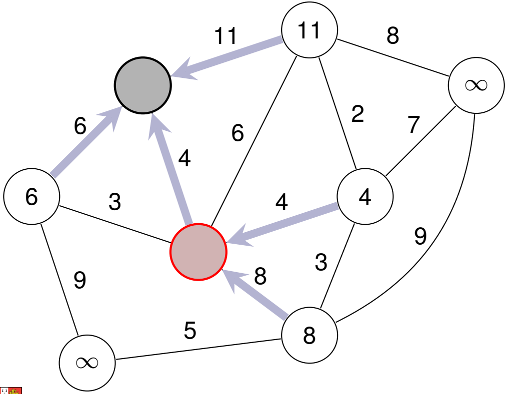

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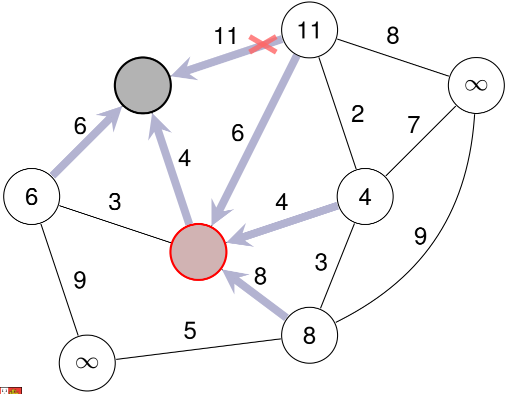

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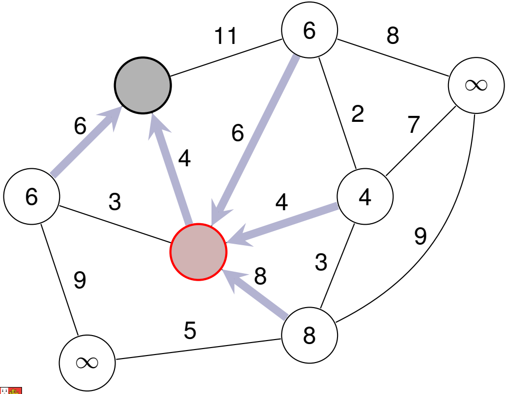

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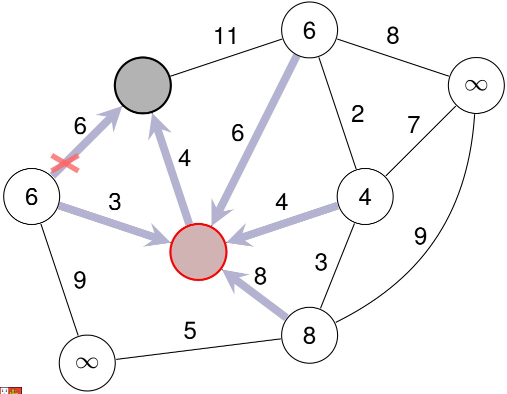

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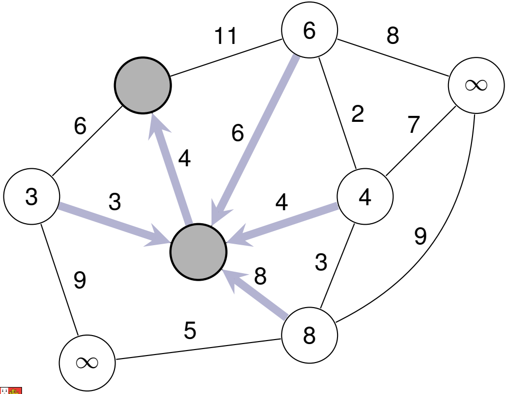

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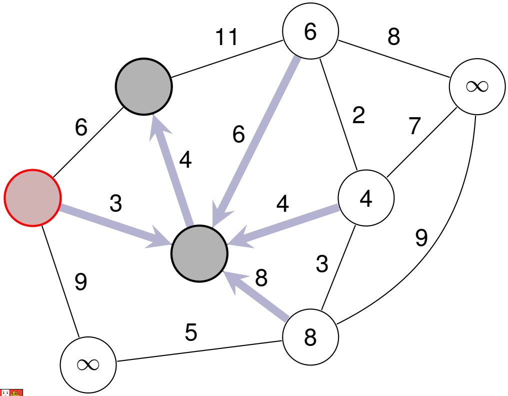

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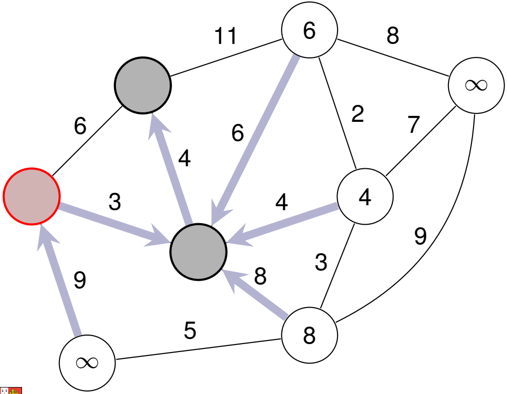

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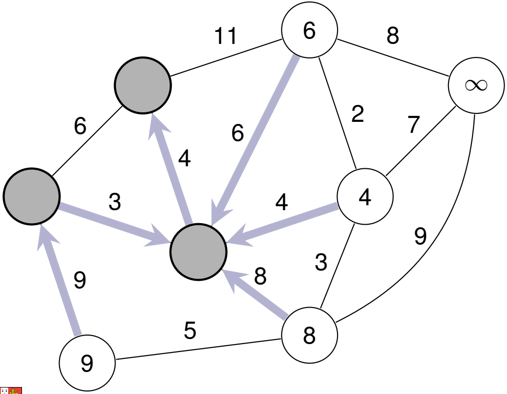

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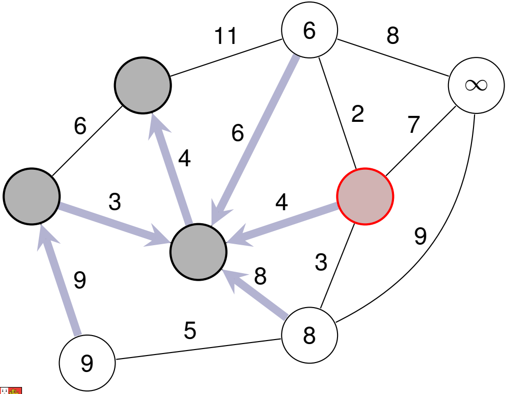

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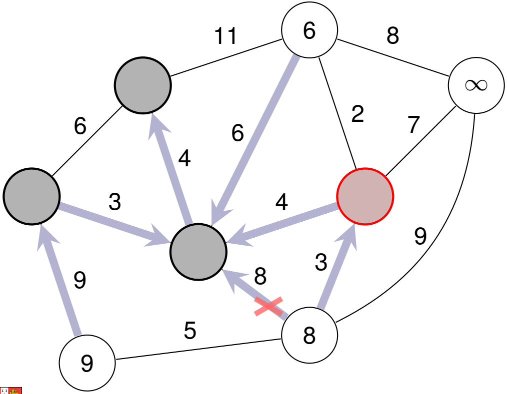

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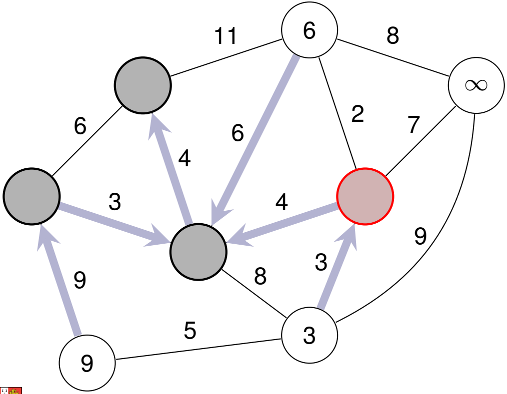

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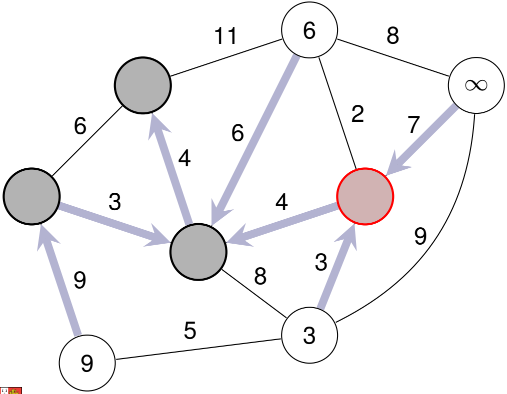

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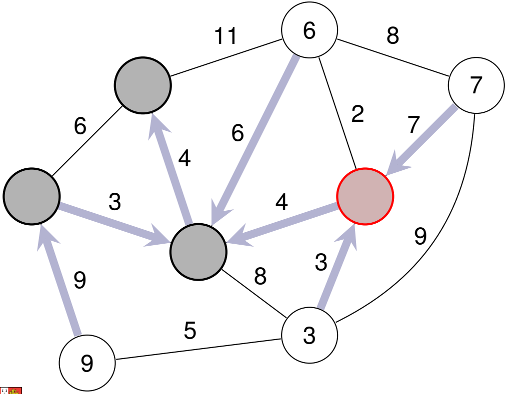

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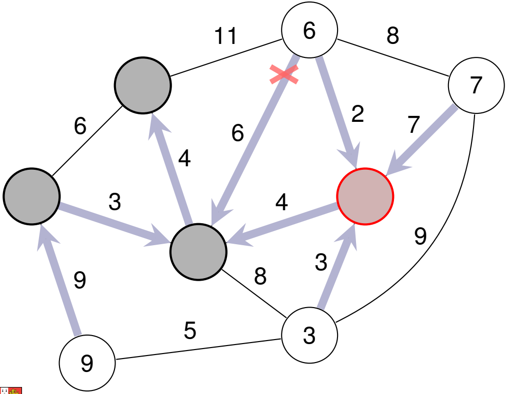

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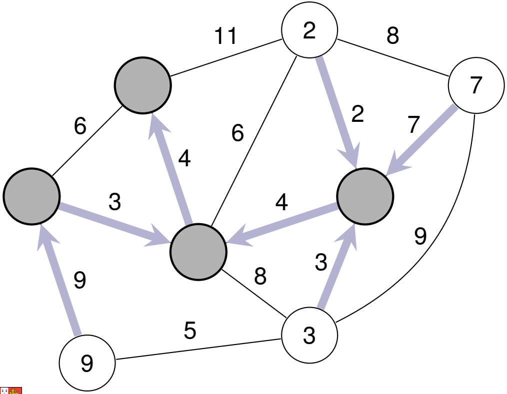

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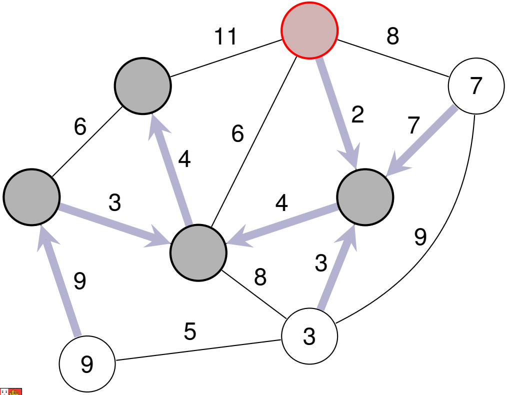

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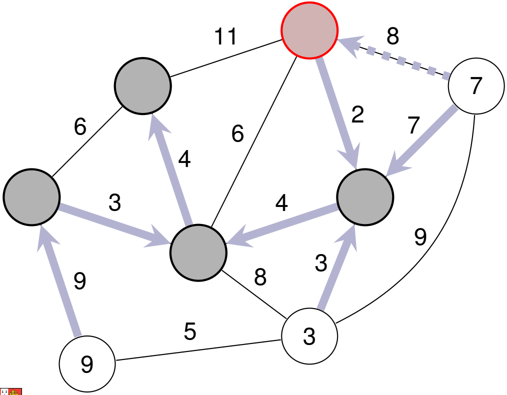

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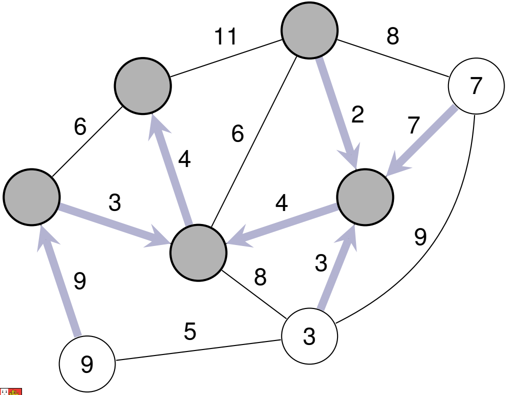

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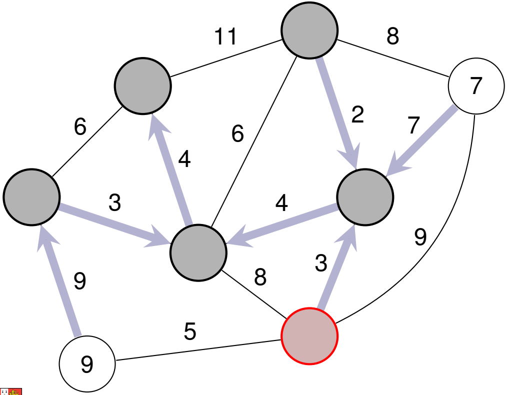

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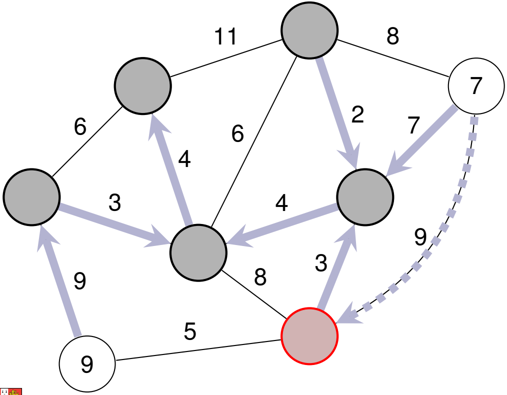

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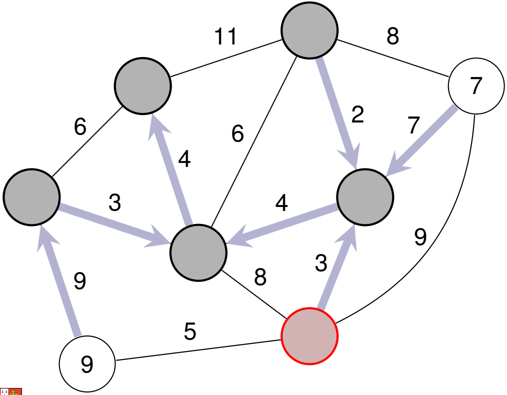

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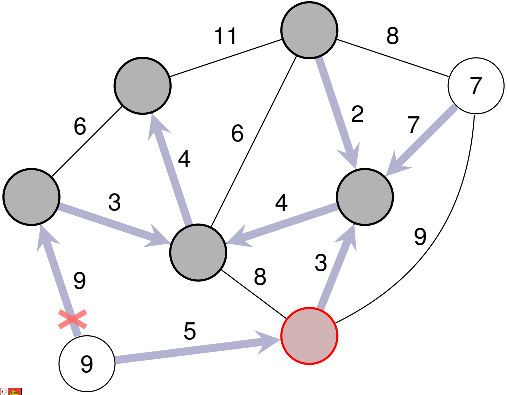

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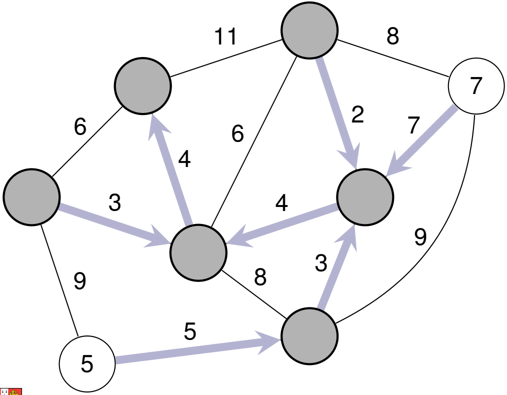

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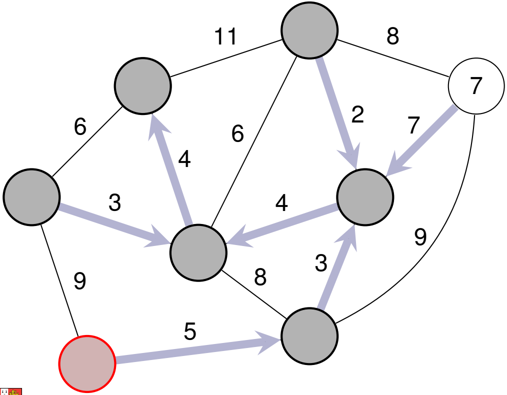

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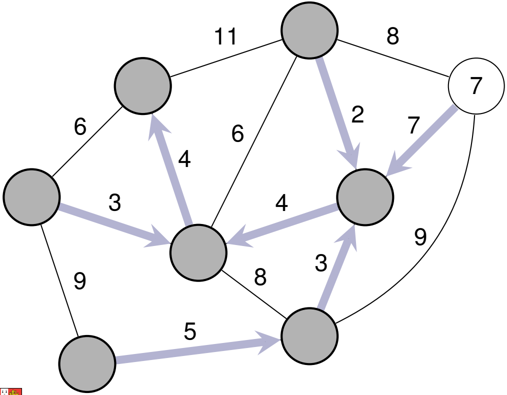

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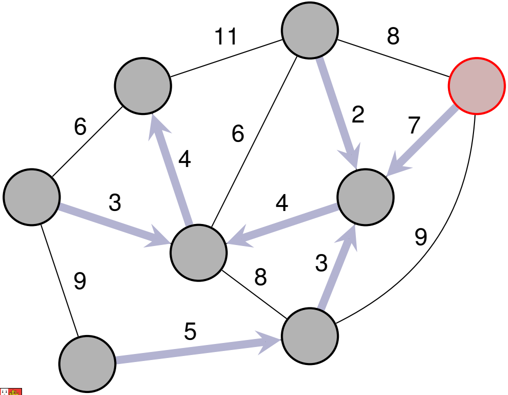

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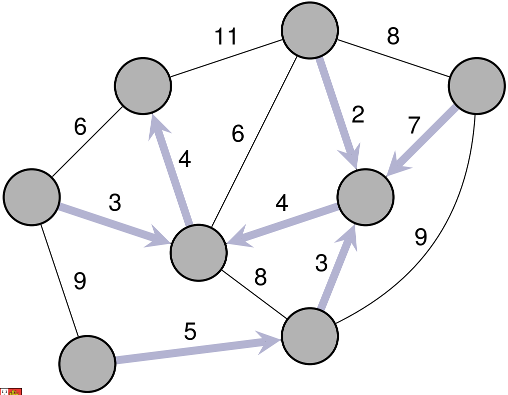

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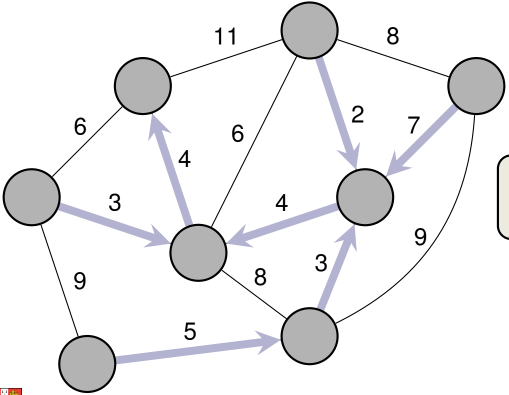

Final MST is given (implicitly) by the pointers!

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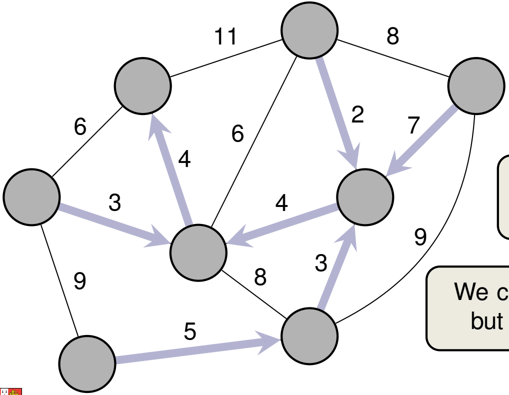

Final MST is given (implicitly) by the pointers!

We computed same MST as Kruskal, but in a completely different order!

Details of Prim's Algorithm

```
def prim(G,r)
    Apply Prim’s Algorithm to graph G and root r
    Return result implicitly by modifying G:
    MST induced by the .predecessor fields

Q = MinPriorityQueue()
for v in G.vertices():
    v.predecessor = None
    if v == r:
        v.key = 0
    else:
        v.key = Infinity
    Q.insert(v)

while not Q.isEmpty():
    u = Q.extractMin()
    for v in u.adjacent():
        w = G.weightOfEdge(u,v)
        if Q.hasItem(v) and w < v.key:
            v.predecessor = u
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Time Complexity

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- invokes disjoint set data structure
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Prim's Algorithm

- Gradually extends a tree into a MST by adding incident edges
- invokes Fibonacci heaps (priority queue)
- Runtime $\mathcal{O}(V \log V + E)$

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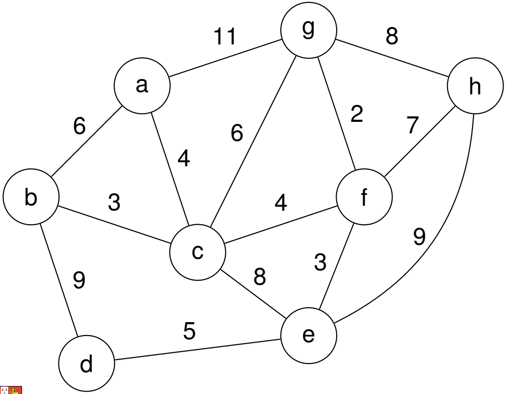

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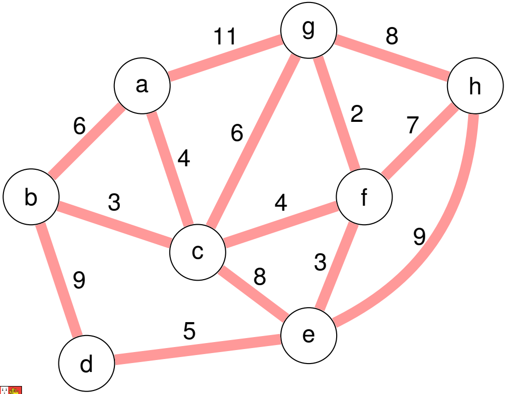

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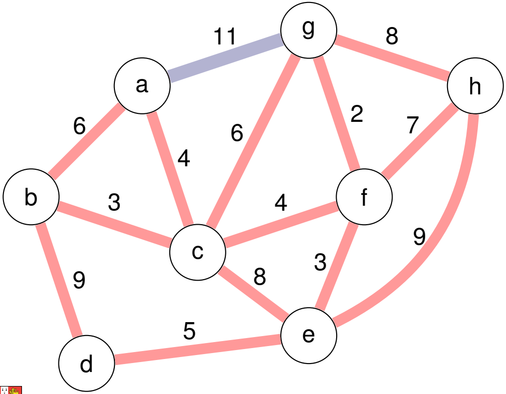

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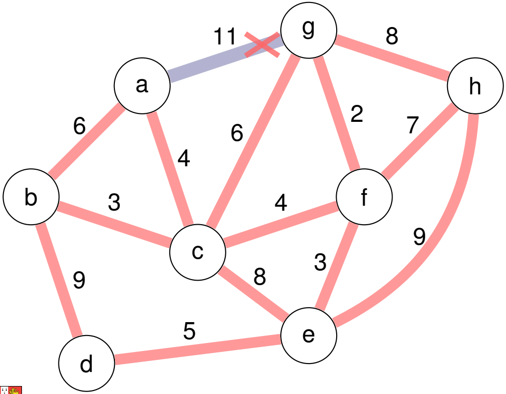

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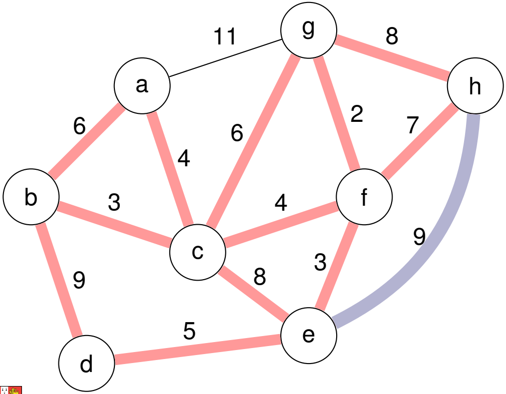

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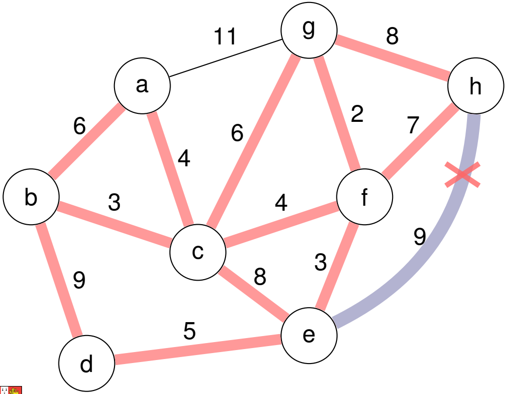

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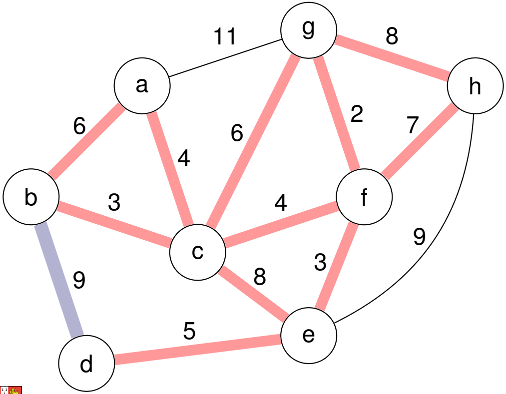

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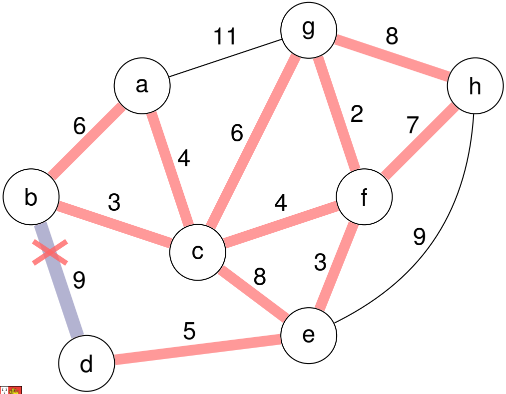

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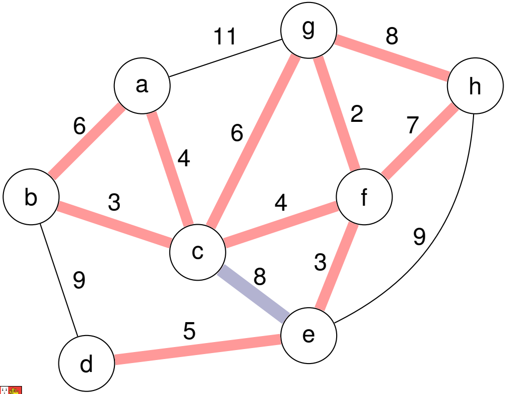

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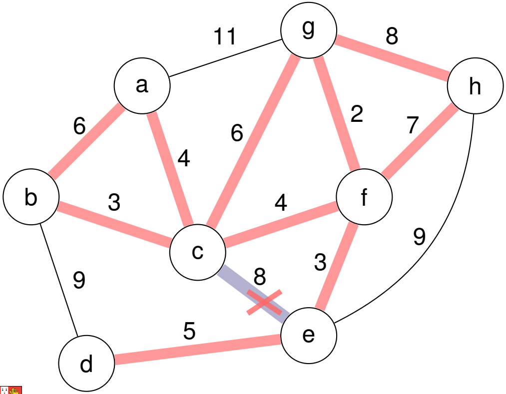

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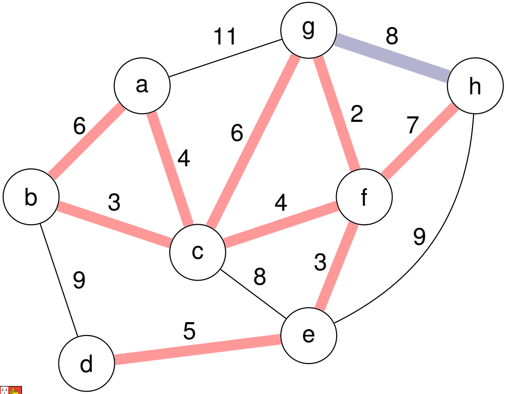

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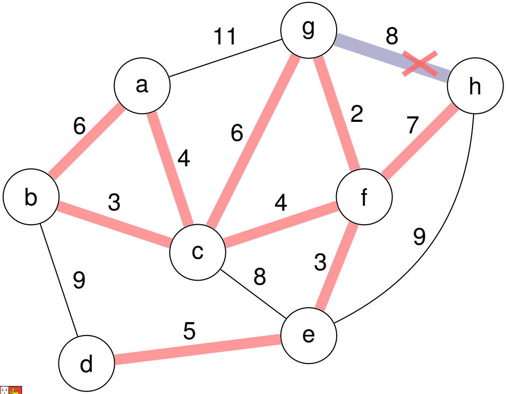

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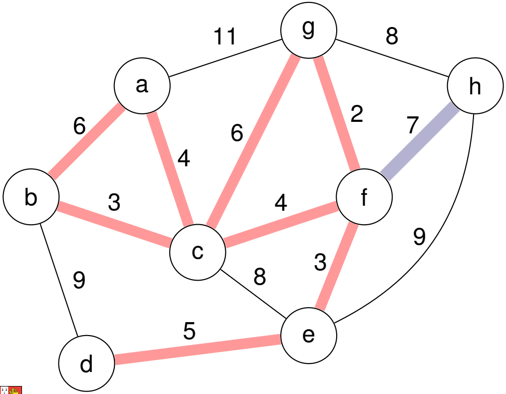

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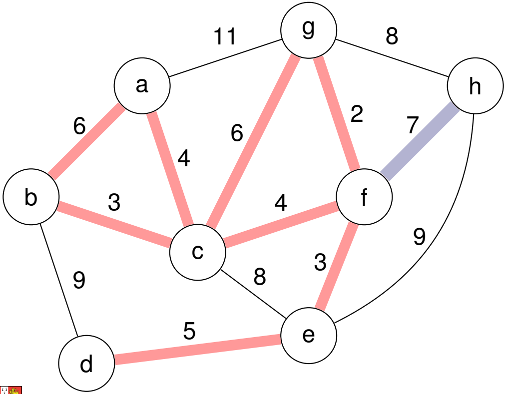

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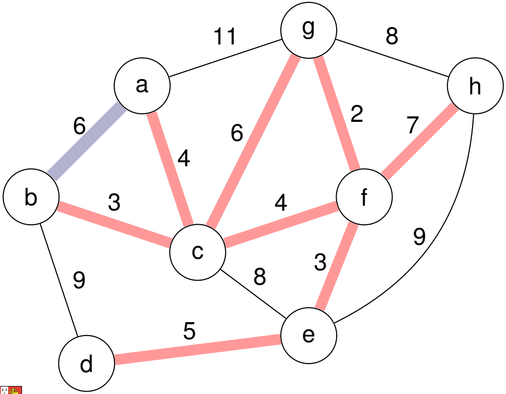

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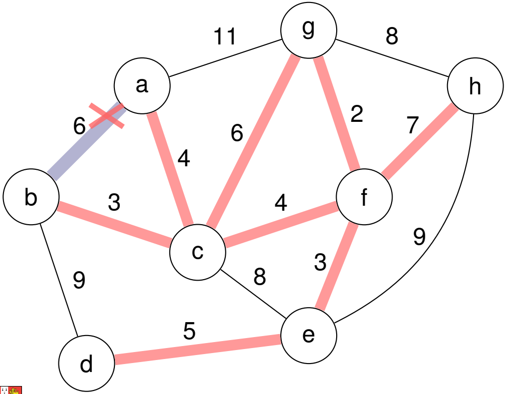

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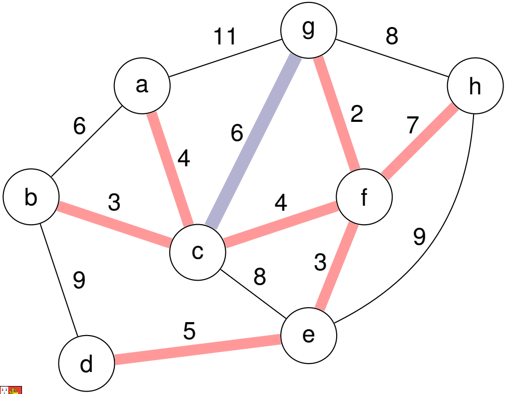

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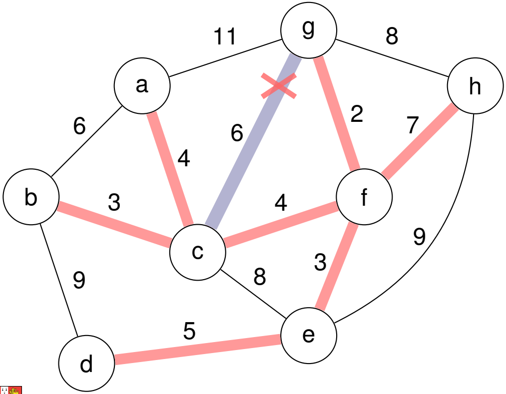

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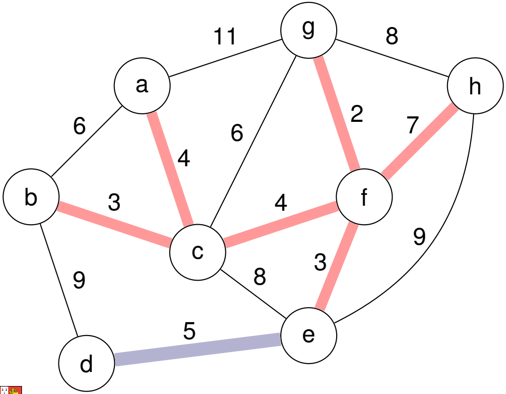

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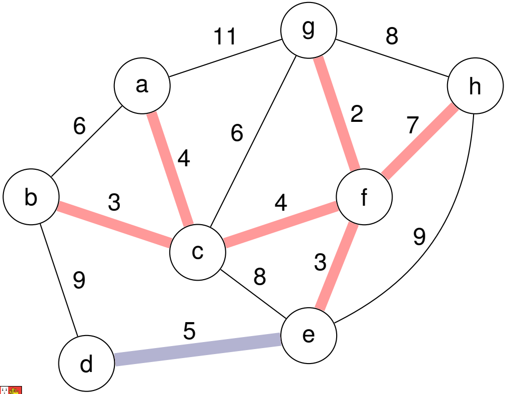

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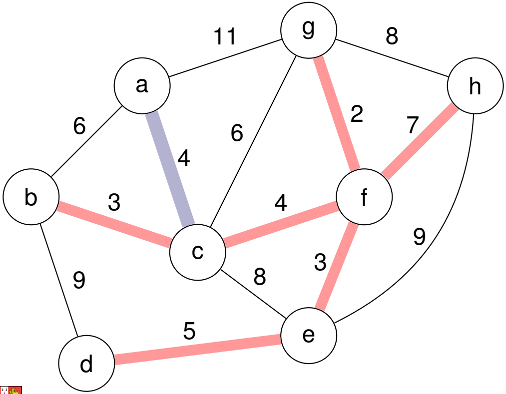

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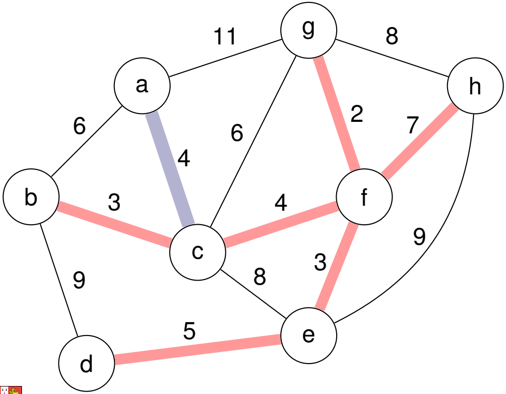

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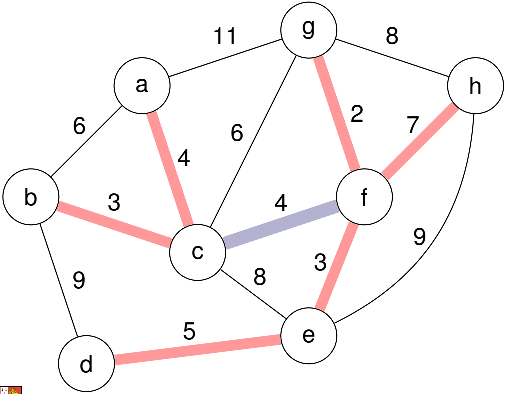

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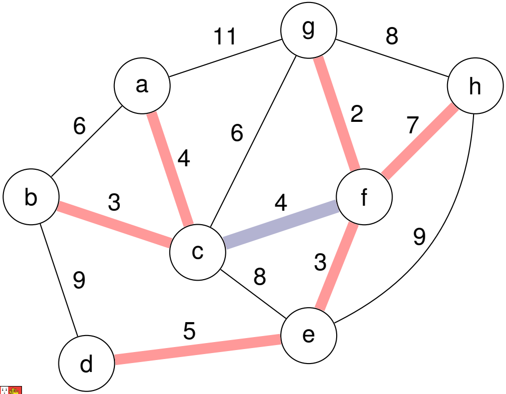

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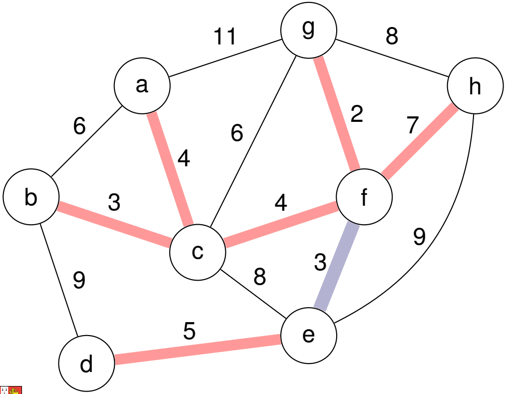

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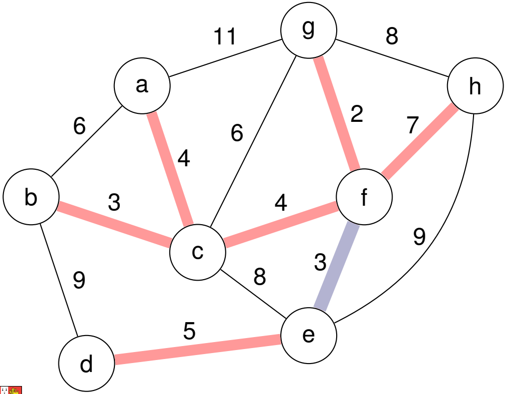

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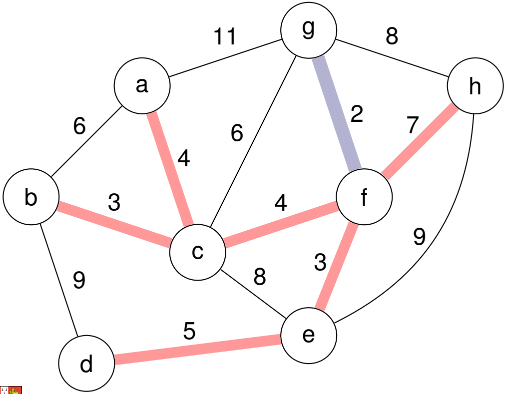

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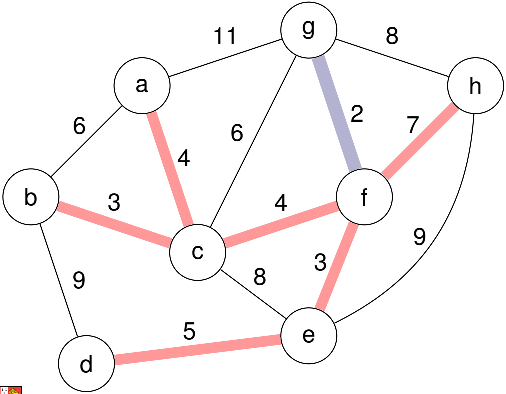

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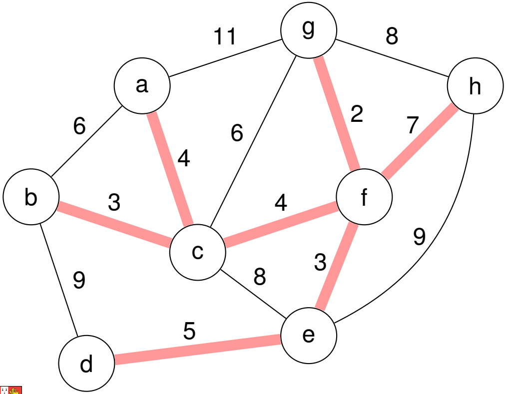

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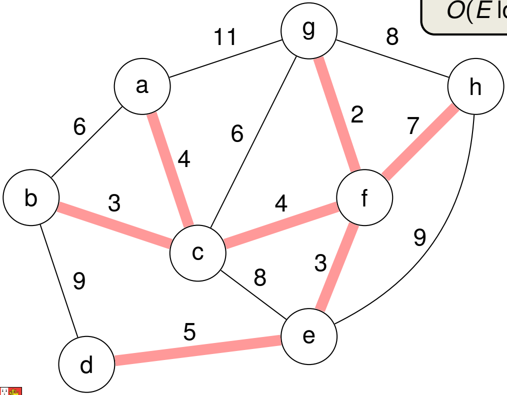
Can be implemented in time $O(E \log V (\log\log V)^3)$. [Thorup, 2000]



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Pettie, Ramachandran, JACM'2002

- deterministic MST algorithm with asymptotically optimal runtime
- however, the runtime itself is not known...